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The Inversion Relation Method for Some Two-Dimensional Exactly Solved Models in Lattice Statistics

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The partition-functions-per-site κ of several two-dimensional models (notably the eight-vertex, self-dual Potts and hard-hexagon models) can be easily obtained by using an inversion relation for local transfer matrices, together with symmetry and analyticity properties. This technique is discussed, the analyticity properties compared, and some equivalences (and nonequivalences) pointed out. In particular, the critical hard-hexagon model is found to have the same κ as the self-dual q-state Potts model, with $q = (3 + \sqrt{5})/2 = 2.618\ldots$. The Temperley–Lieb equivalence between the Potts and six-vertex models is found to fail in certain nonphysical antiferromagnetic cases.

KEY WORDS: Statistical mechanics; lattice statistics; eight-vertex, Ising, Potts, and hard-hexagon models; star–triangle and inversion relations; complete integrability; factorizability.

1. INTRODUCTION

There are a few two-dimensional lattice models in statistical mechanics that have been solved exactly, in the sense that their partition function per site κ has been calculated in the limit of an infinitely large lattice. In particular, there are the zero-field eight-vertex (8V) model[(1)] (which includes the Ising, three-spin, and zero-field six-vertex models as special cases); the self-dual q-state Potts (SDP) model[(2)]; the generalized hard-hexagon (GHH) model[(3)]; and the multistate vertex models of Stroganov[(4)] and Schultz.[(5)] (I shall often drop the adjectives "zero-field" and "generalized.") All these models satisfy the star–triangle relation.[(6)]

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Recently it has been realized[6,7] that for these models κ can be calculated quite easily by using an inversion relation for the local transfer matrices, together with the appropriate analyticity properties of κ. Indeed this is basically the technique used by Stroganov and Schultz. It is also the way the GHH model was originally solved,[3] though a more conventional method has since been used,[8] giving the same results.

Here I show that this inversion relation (and the star–triangle relation) is the same for the 8V, SDP, and GHH models (with appropriate normalization and notation). Provided their relevant analyticity properties are the same, they must therefore have the same partition function per site κ. In this sense I show that the ferromagnetic SDP model is equivalent to a six-vertex model, but the equivalence does not always extend to the antiferromagnetic case. I also show that the critical hard-hexagon model is equivalent to the SDP model with $q = (3 + \sqrt{5})/2 = 2.618\ldots$; and that the hard-hexagon model in its regimes I, III, and IV (see Ref. 3) is equivalent to an eight-vertex model.

The equivalence between the Potts and six-vertex models is not new, having been established by Temperley and Lieb.[9] Here I consider the anisotropic square lattice q-state Potts model, with interaction coefficients K, L satisfying the self-duality condition (3.32). The Temperley–Lieb equivalence is certainly correct for the ferromagnetic model, when K and L are positive real, but in Section 5 I show (guided by the Ising case $q = 2$) that it fails (due to sensitivity to boundary conditions) if e^K or e^L have real part less than $1 - \frac{1}{2}q$.

Such cases are unphysical, in that K and L are not both real (for $q > 1$), but they are still relevant to the dependence of the partition function on K and L. [Also, the critical point of the true hard-hexagon model is equivalent to such a nonphysical SDP model, with $q = (3 + \sqrt{5})/2$, $e^K = 1 - q$, and $e^L = 0$.] I obtain expressions for κ in these cases, provided that $1 < q \leqslant 4$, but I do not see how to extend these to $0 < q < 1$ or to $q > 4$.

I have not seen this limitation of the Temperley–Lieb equivalence previously discussed, but some evidence that the nonferromagnetic cases may be difficult is afforded by the fact that the critical point does not then correspond to the self-dual point.[10]

In Sections 2–5 I focus attention on the six-vertex, SDP, and critical hard-hexagon models. This has the advantage that the Temperley–Lieb equivalence can be discussed, showing how it is related to the star–triangle relation and a linearity property of the Boltzmann weights. It also avoids the need to introduce elliptic functions. In Section 6 I extend the relevant results to the eight-vertex and hard-hexagon models.

I do not consider the Stroganov and Schultz models explicitly, but these satisfy the star–triangle relation and appear to have the linearity property required in Section 2, so they should fit into the discussion of Sections 2, 4, and 5. Unless they have unexpected analyticity properties, they should therefore be equivalent to a six-vertex and/or SDP model. It would be interesting to check this directly.

Note that here I use the word "equivalent" to mean simply that two zero-field models have the same κ for certain values of their parameters, and possibly that their local transfer matrices belong to the same algebra. This is a very weak form of equivalence: it does not imply that their order parameters are the same, so it does not contradict the finding that the critical exponent δ is 15 for the eight-vertex model, 14 for the hard-hexagon model. Nor does it contradict the argument[11,30] that the critical behavior of the hard-hexagon model should be that of the Potts model with $q = 3$, rather than $q = (3 + \sqrt{5})/2$.

2. STAR–TRIANGLE RELATION

For a very large class of square-lattice models in statistical mechanics, the partition function can be written as

$$Z = \mathrm{Tr}(VW)^m \tag{2.1}$$

where

$$\begin{aligned} V &= X_1 X_3 X_5 \ldots X_{n-1} \\ W &= X_2 X_4 X_6 \ldots X_n \end{aligned} \tag{2.2}$$

n being even. For instance, for the Ising and Potts models on a lattice $\mathfrak{L}$ of m rows and $n/2$ columns, X_{2j-1} can be taken to be the local transfer matrix that adds to the lattice a vertical edge in column j; and X_{2j} to be the matrix that adds a horizontal edge between columns j and $j+1$. For the Ising model these matrices are of dimension $2^{n/2}$; for the q-state Potts model they are of dimension $q^{n/2}$.

For the vertex models, expressed in terms of placing arrows on the edges of the lattice $\mathfrak{L}$, (2.1) is the partition function of a square lattice that has been turned through 45°, each X_j corresponding to adding a site (or vertex) to the lattice. For the interactions-round-a-face, or "IRF," models,[6] we use the dual of the previous lattice: each X_j then corresponds to adding a face to the lattice $\mathfrak{L}$. The eight-vertex and six-vertex models can be expressed either as vertex models,[12] or as IRF models.[13] The hard-hexagon model is an IRF model.[3,6]

Let

$$N = mn \tag{2.3}$$

be the total number of X_j's in (2.1). We are interested in calculating

$$\kappa = \lim_{N\to\infty} Z^{1/N} \tag{2.4}$$

where the limit is to be taken through large values of both m and n. For the Potts model this is the partition function per edge, while for the vertex and IRF models κ is the partition function per site.

The nonzero elements of each matrix X_j are the Boltzmann weights of the corresponding edge, site, or face of $\mathfrak{L}$. Let X_j' be another matrix, formed from X_j by merely varying these weights. Then X_i and X_j' commute if i and j are nonadjacent columns of $\mathfrak{L}$, i.e.,

$$X_i X_j' = X_j' X_i \qquad \text{for} \quad |i-j| \geqslant 2 \tag{2.5}$$

Let C_t be the total class of matrices $X_1, \ldots, X_n$ that can be formed by varying the Boltzmann weights in this way. Then the exactly solved models listed in Section 1 have the property that there is a subclass C of C_t such that if X_j and X_j' are members of C, then there exists X_j'' (also a member) such that

$$X_j X_{j+1}' X_j'' = X_{j+1}'' X_j' X_{j+1} \tag{2.6}$$

for $j = 1, \ldots, n$. For the Ising model this is the star–triangle relation,[14–16] so it is convenient to give it this name in general. The relation also occurs in field theory, as I remark at the end of this section.

The significance of (2.6) is that it implies that the diagonal-to-diagonal transfer matrices commute.[1,17] A proper discussion of this point involves the cyclic boundary conditions. Let me merely remark here that if

$$T = X_1 X_2 \ldots X_n, \qquad T' = X_1' X_2' \ldots X_n' \tag{2.7}$$

then (2.6) and (2.5) imply the "quasicommutation" relation

$$TT'A_n = A_1 T'T \tag{2.8}$$

where

$$\begin{aligned} A_1 &= X_1 X_1'' X_1'^{-1} \\ A_n &= X_n'^{-1} X_n'' X_n \end{aligned} \tag{2.9}$$

Apart from boundary terms, T and T' are the diagonal-to-diagonal transfer matrices of $\mathfrak{L}$. When these terms are properly included (first turning $\mathfrak{L}$ through 45°, so that T and T' become the usual row-to-row transfer matrices), then (2.8) becomes simply $TT' = T'T$.

2.1. Temperley–Lieb Operators

The six-vertex, self-dual Potts and critical hard-hexagon models can all be arranged to have the property that if X_j is a given member of C, then all other members are of the form

$$X_j' = \alpha' I + \beta' X_j, \qquad j = 1, \ldots, n \tag{2.10}$$

where α' and β' are scalars, independent of j, and I is the identity matrix. In this sense the class C is linear, any linear combination of members being a member.

Given any values of α' and β' it follows that there exist scalars α'' and β'' such that (2.6) is satisfied, with

$$X_j'' = \alpha'' I + \beta'' X_j, \qquad j = 1, \ldots, n \tag{2.11}$$

Substituting these expressions into the star–triangle relation (2.6), we obtain

$$\alpha'\alpha''(X_j - X_{j+1}) + \alpha'\beta''(X_j^2 - X_{j+1}^2) + \beta'\beta''(X_jX_{j+1}X_j - X_{j+1}X_jX_{j+1}) = 0 \tag{2.12}$$

Looking at some particular nonzero element of this matrix equation, it follows that α'' and β'' are related to α' and β' by an homogeneous equation of the form

$$a\alpha'\alpha'' + b\alpha'\beta'' + c\beta'\beta'' = 0 \tag{2.13}$$

where a, b, c are fixed constants. We can use (2.13) to express α'', β'' in (2.12) in terms of α' and β'. The resulting equation has to be an identity, true for all values of α', β', which gives just two independent equations:

$$aX_j^2 - bX_j = aX_{j+1}^2 - bX_{j+1} \tag{2.14a}$$

$$aX_jX_{j+1}X_j - cX_j = aX_{j+1}X_jX_{j+1} - cX_{j+1} \tag{2.14b}$$

for $j = 1, \ldots, n$.

Let $L = aX_j^2 - bX_j$; then from (2.14a) it is independent of j, so it must commute with all of $X_1, \ldots, X_n$. It is therefore a "quantum number" of the matrices, and we can focus attention on a representation in which it has some particular value l, i.e.,

$$aX_j^2 - bX_j = lI \tag{2.15}$$

where l is a scalar. If we define a set of matrices $U_1, \ldots, U_n$ (belonging to C) by

$$X_j = \rho(I + xU_j) \tag{2.16}$$

then it follows that we can in general choose the scalars ρ and x so that (2.15) and (2.14b) become

$$U_j^2 = q^{1/2}U_j \tag{2.17a}$$

$$U_jU_{j+1}U_j - U_j = U_{j+1}U_jU_{j+1} - U_{j+1} \tag{2.17b}$$

where $j = 1, \ldots, n$ and q is a scalar.

Let S_j be either side of (2.17b). Then it is readily verified that

$$S_j U_j = U_j S_j = q^{1/2} S_j$$
$$S_j U_{j+1} = U_{j+1} S_j = q^{1/2} S_j \tag{2.18a}$$

and

$$S_j^2 = q^{1/2}(q-1) S_j \tag{2.18b}$$

If we consider one particular value of j, and look at a representation in which S_j is diagonal, then these equations imply that S_j, U_j, U_{j+1} all break up into diagonal blocks. In one block $S_j = 0$; in the other block $S_j = q^{1/2}(q-1)I$ and $U_j = U_{j+1} = q^{1/2} I$. We can restrict our attention to the former, in which case we can set $S_j = 0$.

I would like to be able to generalize this argument simultaneously to all values of j, but have not been able to do so using (2.17) only. However, in fact it is true for the six-vertex, SDP, and critical hard-hexagon models that $U_1, \ldots, U_n$ can be chosen so that $S_j = 0$ for $j = 1, \ldots, n$. The relations (2.17) and (2.5) then become

$$U_j^2 = q^{1/2} U_j \tag{2.19a}$$

$$U_j U_{j\pm 1} U_j = U_j \tag{2.19b}$$

$$U_i U_j = U_j U_i, \qquad |i-j| \geqslant 2 \tag{2.19c}$$

for $i, j = 1, \ldots, n$.

Thus the relations (2.19) are "almost" a corollary of the star-triangle relation. Conversely, from (2.10) and (2.11) it is obvious that X_j' and X_j'' can be written similarly to (2.16):

$$X_j' = \rho'(I + x' U_j)$$
$$X_j'' = \rho''(I + x'' U_j) \tag{2.20}$$

where ρ', ρ'', x', x'' are scalars. The relations (2.19) then imply that the star–triangle relation (2.6) is satisfied, provided only that

$$x'' = (x' - x)/(1 + q^{1/2}x + xx') \tag{2.21}$$

[This is Eq. (112) of Ref. 18.]

Up to now I have implicitly used cyclic boundary conditions, so in (2.19) the indices i, j and their difference $i - j$ are to be interpreted modulo n. It is easier to discuss (2.19) if we instead use free boundary conditions, which implies that we take U_1 and U_n to commute for $n \geqslant 3$, and ignore the equation (2.19b) if it involves U_0 or U_{n+1}. There is then no direct relation in (2.19) between U_1 and U_n, other than the commutation property (2.19c).

The relations (2.19) then define a finite-dimensional algebra. For instance, for $n = 2$ any sum of products of U_1, U_2 can be written as

$$a_0 I + a_1 U_1 + a_2 U_2 + a_3 U_1 U_2 + a_4 U_2 U_1 \tag{2.22}$$

where $a_0, \ldots, a_4$ are scalars. For general n, let P be any element of this algebra, and let

$$R = q^{(n'+1)/4} U_1 U_3 U_5 \ldots U_{n'} \tag{2.23}$$

where n' is the largest odd integer not greater than n. Then

$$RPR = \tau(P) R \tag{2.24}$$

where $\tau(P)$ is a scalar that is completely determined by (2.19). For instance, for the simple $n = 2$ case, with P given by (2.22),

$$\tau(P) = q a_0 + q^{3/2} a_1 + q^{1/2} a_2 + q a_3 + q a_4 \tag{2.25}$$

For the Potts model with free boundaries, the partition function is given not by (2.1) and (2.2), but by

$$Z_f = \tau\left[W_f (V_f W_f)^m \right] \tag{2.26}$$

$$\begin{aligned} V_f &= X_1 X_3 X_5 \ldots X_n \\ W_f &= X_2 X_4 X_6 \ldots X_{n-1} \end{aligned} \tag{2.27}$$

where n is now odd. Thus Z_f is determined solely by the equations (2.16), (2.19), and (2.23)–(2.27): it does *not* depend on the representation used for the matrices.

In the next section I shall write down the matrices X_j and U_j for the six-vertex, self-dual Potts and critical hard-hexagon models. They are quite different, being of dimensionality 2^{n+1}, $q^{(n+1)/2}$, $(s_+^{n+4} - s_-^{n+4})/\sqrt{5}$, respectively, where

$$s_\pm = \left(1 \pm \sqrt{5}\right)/2 \tag{2.28}$$

Even so, they satisfy (2.16) and (2.19). If they have the same values of ρ, x, and q, then it follows that they must have the same partition function Z_f. If we make the usual assumption (which I shall question) that the limit (2.4) is independent of boundary conditions, then κ must be the same for all three models.

These considerations can be extended to the case when ρ and x in (2.16) depend on whether j is even or odd (though the resulting problem has not in general been solved). Temperley and Lieb[9] thus showed that the general square-lattice Potts model is equivalent to a staggered (i.e., alternating) six-vertex model. This equivalence was later established graphically, using the free-boundary conditions of this paper.[19]

2.2. Homogeneity and Symmetry

Return to regarding ρ and x as independent of j. Then Z and κ are functions of ρ, x, and q, so we can write κ as $\kappa(\rho, x, q)$. From (2.4), (2.1), and (2.16) it is obvious that κ is homogeneous and linear in ρ, i.e.,

$$\kappa(\rho, x, q) = \rho\,\kappa(1, x, q) \tag{2.29}$$

Also, for the specified models listed in the next section, rotating the lattice $\mathfrak{L}$ through 90° is equivalent to replacing the right-hand side of (2.16) by $\rho(xI + U_j)$, where these new U_j also satisfy (2.19), with the original value of q. It follows that κ must satisfy the symmetry relation

$$\kappa(\rho, x, q) = \kappa(x\rho, x^{-1}, q) \tag{2.30}$$

2.3. Inversion Relation

From (2.16) and (2.19a), inverting X_j is equivalent to the transformation

$$\rho \to \rho^{-1}, \qquad x \to -x/(1 + q^{1/2}x) \tag{2.31}$$

When $x = 0$, then $X_j = \rho I$ and it is obvious from (2.1)–(2.4) that $\kappa = \rho$. By expanding about the point $x = 0$, I have argued[6] that κ can be analytically continued through $x = 0$, and that the resulting analytic function satisfies the same inversion relation as X_j, i.e.,

$$\kappa(\rho, x, q)\kappa_{\mathrm{ac}}\left(\rho^{-1}, \frac{-x}{1 + q^{1/2}x}, q\right) = 1 \tag{2.32}$$

where κ is the true partition function per site on one side of $x = 0$ (say x positive), and κ_{ac} is its analytic continuation to the other side (x negative).

This argument has recently been put on firmer footing.[7] The star–triangle relation (2.6) ensures that the diagonal transfer matrices T commute for different values of x (but the same q). This implies that their eigenvalues are entire functions of x. It follows that κ has a different analytic expression for x positive from that for x negative, that each can be analytically continued across $x = 0$, and that each satisfies (2.32).

The relations (2.29), (2.30), and (2.32), together with some basic analyticity properties of κ, considered as a function of the complex variable x, completely determine κ. The main purpose of this paper is to investigate these properties and the resulting solutions. This is done in Sections 4 and 5, but before finishing this section it is convenient to establish some further notation.

Applying the rotation symmetry (2.30) to (2.32), and using (2.29), gives

$$\kappa(\rho, x, q)\kappa_{\mathrm{ac}}(\rho^{-1}, -x - q^{1/2}, q) = -x(x + q^{1/2}) \tag{2.33}$$

where now κ_{ac} is the analytic continuation of κ through $x = \infty$. The relation (2.32) applies in the neighborhood of $x = 0$; (2.33) applies in the neighborhood of $x = \infty$: I shall therefore refer to $x = 0$ and $x = \infty$ as *inversion points*.

The point $x = 0$ is a fixed point of the transformation (2.31); the other fixed point is $x = -2q^{-1/2}$. The arguments that lead to (2.32) do not apply to this fixed point, since X_j is not here proportional to the identity matrix (though its square is). If κ_{ac} is interpreted as the analytic continuation of κ through this fixed point, then (2.32) is not necessarily true. (This is because the analytic continuation can be a multivalued function of x.) I shall therefore say that $x = -2q^{-1/2}$ is a *virtual inversion point* of (2.32). Similarly, $x = -q^{1/2}/2$ is a virtual inversion point of (2.33).

From now on let us take q and $q^{1/2}$ to be positive real. For $q > 4$ it is convenient to transform variables from ρ, x, q to ρ_0, u, λ by

$$\begin{aligned} q^{1/2} &= 2\cosh\lambda \\ x &= \sinh u/\sinh(\lambda - u) \\ \rho &= \rho_0 \sinh(\lambda - u)/\sinh\lambda \end{aligned} \tag{2.34}$$

where

$$\lambda > 0, \qquad 0 \leqslant \mathrm{Im}(u) < \pi \tag{2.35}$$

Then (2.16) becomes

$$X_j = \rho_0\left[\sinh(\lambda - u) + (\sinh u)U_j\right]/\sinh\lambda \tag{2.36}$$

the inversion points $x = 0, \infty$ become

$$u = 0, \lambda \tag{2.37}$$

and the virtual inversion points $x = -2/q^{1/2}, \ -q^{1/2}/2$ become

$$u = \tfrac{1}{2}i\pi, \tfrac{1}{2}i\pi + \lambda \tag{2.38}$$

Regarding ρ_0 and λ as constants, u as a variable, and writing κ as $\kappa(u)$, the symmetry and inversion relations (2.30), (2.32), (2.33) become, using (2.29),

$$\kappa(u) = \kappa(\lambda - u) \tag{2.39}$$

$$\kappa(u)\kappa_{ac}(-u) = \rho_0^2 \sinh(\lambda - u)\sinh(\lambda + u)/\sinh^2\lambda \tag{2.40a}$$

$$\kappa(u)\kappa_{ac}(2\lambda - u) = \rho_0^2 \sinh u \sinh(2\lambda - u)/\sinh^2\lambda \tag{2.40b}$$

the last two equations being valid in the neighborhood of the inversion points 0, λ, respectively.

For $q < 4$ the parameter λ becomes pure imaginary and it is convenient to replace u, λ by iv, $i\mu$. The equations (2.34)–(2.40) then become

$$\begin{gathered} q^{1/2} = 2\cos\mu, \qquad x = \sin v/\sin(\mu - v) \\ \rho = \rho_0 \sin(\mu - v)/\sin\mu \end{gathered} \tag{2.41}$$

$$0 < \mu < \pi/2, \qquad 0 \leqslant \mathrm{Re}(v) < \pi \tag{2.42}$$

$$X_j = \rho_0\left[\sin(\mu - v) + \sin v U_j\right]/\sin\mu \tag{2.43}$$

$$v = 0, \mu \qquad \text{are inversion points} \tag{2.44}$$

$$v = \frac{\pi}{2}, \frac{\pi}{2} + \mu \qquad \text{are virtual inversion points} \tag{2.45}$$

$$\kappa(v) = \kappa(\mu - v) \tag{2.46}$$

$$\kappa(v)\kappa_{\mathrm{ac}}(-v) = \rho_0^2 \sin(\mu - v)\sin(\mu + v)/\sin^2\mu \tag{2.47a}$$

$$\kappa(v)\kappa_{\mathrm{ac}}(2\mu - v) = \rho_0^2 \sin v \sin(2\mu - v)/\sin^2\mu \tag{2.47b}$$

The case $q = 4$ can be handled by taking the limit $q \to 4^+$ or $q \to 4^-$.

Using the $q > 4$ notation, let u, u', u'' be the values of u corresponding to x, x', x'' in (2.16) and (2.20). Then the condition (2.21) becomes very simple:

$$u'' = u' - u \tag{2.48}$$

(This is closely related to the "transformation to a difference kernel" that occurs in the Bethe ansatz.[20,21]) Still regarding λ and ρ_0 as constants, X_j is a function of u, so the star–triangle relation (2.6) can be written more explicitly as

$$X_j(u)X_{j+1}(u')X_j(u' - u) = X_{j+1}(u' - u)X_j(u')X_{j+1}(u) \tag{2.49}$$

The inversion transformation (2.31) corresponds to the relation

$$X_j(u)X_j(-u) = \left[\rho_0^2 \sinh(\lambda - u)\sinh(\lambda + u)/\sinh^2\lambda\right]I \tag{2.50}$$

Appropriately normalized, the matrix $X_j(u)$ becomes the S matrix of field theory: (2.49) and (2.50) are then the conditions for complete integrability and factorizability.[22]

3. SPECIFIC MODELS

As I remarked in Section 2, the six-vertex, SDP, and critical hard-hexagon models all have local transfer matrices satisfying (2.6), (2.16), and (2.19). In this section I shall define these models and thereby give three explicit representations of the Temperley–Lieb algebra (2.19). In each case I start by defining a more general model (eight-vertex, Potts, and hard-hexagon, respectively). Here I shall use the Ising-like formulation of the vertex models.[13]

It is convenient to start with the very general "interactions-round-a-face" (IRF) model.[6] Take $\mathfrak{L}$ to be the square lattice turned through 45°, with N sites. At each site i place a "spin" σ_i which has some discrete set of allowed values. To each face (i, j, k, l), where i, j, k, l are the four sites round the face, arranged anticlockwise from the bottom, assign a Boltzmann weight $w(\sigma_i, \sigma_j, \sigma_k, \sigma_l)$. The partition function is

$$Z = \sum_{\sigma} \prod_{(i,j,k,l)} w(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \tag{3.1}$$

where the product is over all faces and the sum is over all values of $\sigma_1, \ldots, \sigma_N$. This can be put into the form (2.1), where X_j is a matrix whose rows are labeled by the spin set $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_n\}$, whose columns are labeled by $\boldsymbol{\sigma}' = \{\sigma'_1, \ldots, \sigma'_n\}$, and whose elements are

$$\begin{aligned}(X_j)_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} = {} & \delta(\sigma_1, \sigma'_1) \cdots \delta(\sigma_{j-1}, \sigma'_{j-1}) \\ & \times w(\sigma_j, \sigma_{j+1}, \sigma'_j, \sigma_{j-1}) \delta(\sigma_{j+1}, \sigma'_{j+1}) \cdots \delta(\sigma_n, \sigma'_n)\end{aligned} \tag{3.2}$$

for $j = 1, \ldots, n$. If each σ_i has two values, then X_j is a $2^n \times 2^n$ matrix.

If we use cyclic boundary conditions, as in (2.1) and (2.2), then the indices in (3.2) should be interpreted modulo n. If we use free boundary conditions, as in (2.26) and (2.27), then we should extend the spin sets to $\boldsymbol{\sigma} = \{\sigma_0, \sigma_1, \ldots, \sigma_{n+1}\}$, $\boldsymbol{\sigma}' = \{\sigma'_0, \sigma'_1, \ldots, \sigma'_{n+1}\}$, and introduce extra factors $\delta(\sigma_0, \sigma'_0)$, $\delta(\sigma_{n+1}, \sigma'_{n+1})$ into (3.2).

3.1. Eight-Vertex Model

Let each spin σ_i take the values $+1$ and -1, or simply $+$ and $-$. Then for the eight-vertex model w is defined by

$$w(\alpha, \beta, \gamma, \delta) = w(-\alpha, -\beta, -\gamma, -\delta) \tag{3.3}$$

for $\alpha, \beta, \gamma, \delta = \pm 1$, and

$$\begin{aligned} w(++++) &= \omega_1, & w(+-+-) &= \omega_2 \\ w(+--+) &= \omega_3, & w(++--) &= \omega_4 \\ w(+++-) &= \omega_5, & w(+-++) &= \omega_6 \\ w(++-+) &= \omega_7, & w(-+++) &= \omega_8 \end{aligned} \tag{3.4}$$

The $\omega_1, \ldots, \omega_8$ are "vertex weights." Here I shall consider only the "zero-field" case, when

$$\omega_1 = \omega_2 \quad \text{and} \quad \omega_3 = \omega_4 \tag{3.5}$$

Define a, b, c, d, s, t by

$$\omega_1, \ldots, \omega_8 = a, a, b, b, cs, cs^{-1}, dt, dt^{-1} \tag{3.6}$$

It is easy to see[12] that for cyclic boundary conditions the weights ω_5 and ω_6 must occur in pairs. So must ω_7 and ω_8. Thus Z is independent of s and t, so κ is a function only of a, b, c, d. It is an even function of each variable, i.e.,

$$\kappa(\pm a, \pm b; \pm c, \pm d) = \kappa(a, b; c, d) \tag{3.7}$$

for all independent choices of the signs. It also satisfies the symmetry property

$$\kappa(a, b; c, d) = \kappa(b, a; c, d) \tag{3.8}$$

corresponding to rotating the lattice through 90° and then negating spins in alternate columns.

Six-Vertex Model. The six-vertex model is obtained from the eight-vertex model by setting

$$\omega_7 = \omega_8 = d = 0 \tag{3.9}$$

In this case the model can be solved even in the presence of fields, i.e., when (3.5) is violated.[20,21,23] It would be interesting to consider this case from the present viewpoint, but I shall only consider the zero-field case.

Two parameters that occur in the solution of this model are Δ and λ, where

$$\Delta = (a^2 + b^2 - c^2)/(2ab) = -\cosh\lambda \tag{3.10}$$

We can choose s in any convenient way. It follows that we can define ρ and x so that

$$\omega_1, \ldots, \omega_6 = \rho(1, 1, x, x, 1 + xe^{\lambda}, 1 + xe^{-\lambda}) \tag{3.11}$$

From (3.2) and (3.4), it follows that X_j is then of the form (2.16), where U_j depends on λ, but not on ρ or x. The elements of U_j are given by the right-hand side of (3.2), with the function w replaced by

$$\tfrac{1}{2}(1 - \sigma_{j-1}\sigma_{j+1})\exp\left[\tfrac{1}{2}\lambda\sigma_{j+1}(\sigma_j + \sigma_j')\right] \tag{3.12}$$

We can verify directly that the relations (2.19) are satisfied, with

$$q^{1/2} = 2\cosh\lambda = -2\Delta \tag{3.13}$$

The λ herein is therefore the same as that in (2.34)–(2.40). Defining u and ρ_0 by (2.34), from (3.6) and (3.11) we obtain

$$[a, b, c] = \rho_0 \operatorname{cosech}\lambda\left[\sinh(\lambda - u), \sinh u, \sinh\lambda\right] \tag{3.14}$$

Alternatively, using the μ, v of (2.41)–(2.48)

$$[a, b, c] = \rho_0 \operatorname{cosec}\mu\left[\sin(\mu - v), \sin v, \sin\mu\right] \tag{3.15}$$

With these identifications, all the relations of Section 2 apply to the six-vertex model. By using the evenness property (3.7), for real values of a, b, c we can always arrange (if necessary by taking b negative) that

$$\Delta < 0, \qquad q^{1/2} > 0 \tag{3.16}$$

so the restrictions (2.35) or (2.42) can always be satisfied.

3.2. Generalized Hard-Hexagon Model

This is also an IRF model, so Z and X_j are again given by (3.1) and (3.2). Now, however, each σ_i takes the values 0 or 1, and

$$\begin{aligned} w(\alpha, \beta, \gamma, \delta) &= m z^{(\alpha+\beta+\gamma+\delta)/4} e^{L\alpha\gamma + M\beta\delta} t^{-\alpha+\beta-\gamma+\delta} \\ &\qquad \text{if} \quad \alpha\beta = \beta\gamma = \gamma\delta = \delta\alpha = 0 \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{3.17}$$

Here m is a trivial normalization factor, z is the activity, L and M are interaction coefficients, and t is a disposable parameter that cancels out of the partition function. The parameters z, L, M must satisfy the restriction

$$z = (1 - e^{-L})(1 - e^{-M})/(e^{L+M} - e^{L} - e^{M}) \tag{3.18}$$

This restriction is necessary for the star–triangle relation (2.6) to have nontrivial solutions and for the model to be solvable by existing methods: without it we have the general hard-square lattice gas with diagonal interactions. The triangular lattice gas (i.e., the proper hard-hexagon model) is the limiting case $L \to 0$, $M \to -\infty$.

There are five distinct nonzero Boltzmann weights:

$$\begin{aligned} \omega_1 &= w(0000) = m \\ \omega_2 &= w(1000) = w(0010) = m z^{1/4} t^{-1} \\ \omega_3 &= w(0100) = w(0001) = m z^{1/4} t \\ \omega_4 &= w(1010) = m z^{1/2} t^{-2} e^{L} \\ \omega_5 &= w(0101) = m z^{1/2} t^{2} e^{M} \end{aligned} \tag{3.19}$$

and it is useful to define a parameter

$$\Delta_h = z^{-1/2}(1 - z e^{L+M}) = (\omega_1^2 - \omega_4\omega_5)/\omega_2\omega_3 \tag{3.20}$$

Critical Case. This model has been solved exactly.[3] It is critical when

$$\Delta_h^{-2} = \left[\tfrac{1}{2}(\sqrt{5} + 1)\right]^5 = \tfrac{1}{2}(11 + 5\sqrt{5}) \tag{3.21}$$

In this case we can choose t, and parameters ρ_0, μ, v, so that

$$\begin{aligned}
\omega_1 &= \rho_0 \sin(v + 2\mu)/\sin 2\mu \\
\omega_2 &= \rho_0 \sin v/[\sin \mu \sin 2\mu]^{1/2} \\
\omega_3 &= \rho_0 \sin(\mu - v)/\sin \mu \\
\omega_4 &= \rho_0 \sin(2\mu - v)/\sin 2\mu \\
\omega_5 &= \rho_0 \sin(\mu + v)/\sin \mu
\end{aligned} \tag{3.22}$$

where

$$\mu = \pi/5 \tag{3.23}$$

This is the parametrization of Eqs. (41) and (42) of Ref. 3, and of Eq. (2.12) of Ref. 8, with q, u therein replaced by 0, v. The "physical" cases are when ρ_0 is positive and $-\mu < v < \mu$: the weights $\omega_1, \ldots, \omega_5$ are then positive (except possibly for ω_2, but negating ω_2 leaves Z unchanged, so we can always map this to a physical case, and vice versa).

Defining q, x, ρ by (2.41), it follows that

$$\begin{aligned}
&\omega_1 = \rho(1 + x), \qquad && \omega_2 = \rho q^{-1/4} x \\
&\omega_3 = \rho, && \omega_4 = \rho(1 + q^{-1/2} x) \\
&\omega_5 = \rho(1 + q^{1/2} x)
\end{aligned} \tag{3.24}$$

where

$$q^{1/2} = 2\cos \mu, \qquad q = \tfrac{1}{2}(3 + \sqrt{5}) = 2.618 \ldots \tag{3.25}$$

The definition (3.17) ensures that the only states which contribute to Z are those in which $\sigma_i \sigma_j = 0$ for all adjacent sites i and j (i.e., no two particles can be adjacent). We can therefore restrict the spin set $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_n\}$ in (3.2) to satisfy the restriction

$$\sigma_j \sigma_{j+1} = 0, \qquad j = 1, \ldots, n \tag{3.26}$$

and similarly for $\boldsymbol{\sigma}'$. From (3.19) and (3.24), it then follows that X_j is of the form (2.16). The elements of U_j are given by (3.2), with w replaced by

$$\delta(\sigma_{j-1}, \sigma_{j+1}) q^{(-\sigma_j + \sigma_{j+1} - \sigma_j' + \sigma_{j-1})/4} \tag{3.27}$$

These matrices U_j satisfy (3.19), so the relations of Section 2 apply also to this model, the parameters ρ_0, μ, v, ρ, x, q herein being the same of those of Section 2. Note that q is restricted to the value given in (3.25), whereas in the six-vertex and SDP models q can take any positive real value. (There is also a nonphysical critical case of the hard-hexagon model, corresponding to $\Delta_h^{-2} = \frac{1}{2}(11 - 5\sqrt{5})$, $\mu = 2\pi/5$, $q = \frac{1}{2}(3 - \sqrt{5}) = 0.382 \ldots$, but I shall only consider the physical case.)

3.3. Potts Model

For the Potts model, take $\mathfrak{L}$ to be the usual square lattice, with m rows, $n/2$ columns, and $N/2$ sites. At each site i place a spin σ_i, with values $1, \ldots, q$. The partition function is

$$Z = R^N \sum_{\sigma} \exp\left[K \sum_{(ij)} \delta(\sigma_i, \sigma_j) + L \sum_{(kl)} \delta(\sigma_k, \sigma_l) \right] \tag{3.28}$$

where the first summation inside the exponential is over all horizontal edges (i, j) of $\mathfrak{L}$, the second is over all vertical edges (k, l), K and L are interaction coefficients, the outer sum is over all states of all the spins, and R is merely a normalization factor per edge. Normally we would take $R = 1$.

Kasteleyn and Fortuin[24] and Baxter, Kelland, and Wu[19] showed that Z could be written as

$$Z = R^N \sum_{G} q^C (e^K - 1)^r (e^L - 1)^s \tag{3.29}$$

where the sum is over all graphs G on $\mathfrak{L}$, C is the number of connected components (including isolated sites) in G, r is the number of horizontal bonds, and s the number of vertical bonds. This is a dichromatic polynomial[25] and this form can be used to extend the definition of Z to noninteger values of q.

The edge transfer matrices are $X_1, \ldots, X_n$, as in (2.2) and (2.27). They can be written as

$$\begin{aligned} X_{2j-1} &= Rs^{-1}\left[(e^L - 1) I + q^{1/2} U_{2j-1} \right] \\ X_{2j} &= Rs\left[I + q^{-1/2} (e^K - 1) U_{2j} \right] \end{aligned} \tag{3.30}$$

for $1 \leqslant j \leqslant n/2$. Here s is a parameter that cancels out of the partition function, so we can choose it to suit our convenience. The matrices $U_1, \ldots, U_n$ are q^p by q^p, where $p = n/2$. They have rows labeled by the spin set $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_p\}$, and columns labeled by $\boldsymbol{\sigma}' = \{\sigma_1', \ldots, \sigma_p'\}$, and elements

$$(U_{2j-1})_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} = q^{-1/2} \prod_{k=1}^{p}{}^{*} \, \delta(\sigma_k, \sigma_k') \tag{3.31a}$$

$$(U_{2j})_{\boldsymbol{\sigma}, \boldsymbol{\sigma}'} = q^{1/2} \delta(\sigma_j, \sigma_{j+1}) \prod_{k=1}^{p} \delta(\sigma_k, \sigma_k') \tag{3.31b}$$

the star * in the first product meaning that the term $k = j$ is excluded. Thus U_{2j} is a diagonal matrix; U_{2j-1} is not. For cyclic boundary conditions, $j = 1, \ldots, p$ in (3.31), and the index $j + 1$ is to be interpreted modulo p. For free boundary conditions, n is odd, the definition of p is changed to

$p=(n+1)/2$, (3.31a) holds for $j=1,\ldots,p$, and (3.31b) for $j=1,\ldots,p-1$.

In general the Potts model has not been solved exactly: exceptions are the $q=2$ case, which is the Ising model; the trivial case $q=1$; and the self-dual case, which I discuss in this paper.

Self-Dual Case (The SDP Model). The matrices $U_1,\ldots,U_n$ satisfy the relations (2.19) (they are the third realization found in this section of such matrices). We can put (3.30) into the form (2.16) only if

$$(e^K-1)(e^L-1)=q \tag{3.32}$$

in which case

$$\begin{aligned}\rho &= R(e^L-1)^{1/2}\\ x &= q^{1/2}/(e^L-1)=q^{-1/2}(e^K-1)\end{aligned} \tag{3.33}$$

Equation (3.32) is the condition for the Potts model to be self-dual.[26] In fact, duality corresponds to simply interchanging odd and even indices in (3.30).

To summarize this section: the six-vertex, critical hard-hexagon, and SDP models all have local transfer matrices satisfying (2.16) and (2.19). They therefore all satisfy the star–triangle and inversion relations (2.6), (2.47), and (2.50). In the next two sections I shall discuss and compare their partition functions per site κ.

4. THE FUNCTION κ FOR $q>4$

4.1. Six-Vertex Model

The exact solution of the six-vertex model, with cyclic boundary conditions, has been rigorously obtained[20,21] for positive values of the weights a, b, c, using the theorems of Yang and Yang.[27] The evenness relations (3.7) (with $d=0$) can be used to extend these to all real values of a, b, and c. The parameters q, λ, ρ_0, u are related to a, b, c by (3.10), (3.13), and (3.14). For $q>4$ the parameter λ is positive and u is real. There are three cases.

(i) $0<u<\lambda$: This corresponds to the ordered antiferroelectric phase, such as that of the F model. We have

$$\ln\kappa=\ln\rho_0+2\sum_{n=1}^{\infty}\frac{e^{-2n\lambda}\sinh nu\,\sinh n(\lambda-u)}{n\cosh n\lambda} \tag{4.1a}$$

[note that the symmetry property (2.39) is obviously satisfied].

(ii) $u>\lambda$: This is an ordered ferroelectric KDP-type phase, which is completely frozen; κ is simply equal to the numerically largest Boltzmann weight b, so

$$\kappa = \rho_0 \sinh u / \sinh \lambda \tag{4.1b}$$

(iii) $u < 0$: This case can be obtained from (ii) by rotating $\mathfrak{L}$ through 90°:

$$\kappa = \rho_0 \sinh(\lambda - u)/\sinh \lambda \tag{4.1c}$$

Analyticity and Periodicity of $\kappa(u)$. These results have been rigorously obtained only for real values of u, but they can obviously be analytically continued to complex values of u. From large-λ expansions, it seems that they then give the true partition function per site, provided that case (i) is taken to apply in the domain $0 < \mathrm{Re}(u) < \lambda$; case (ii) in $\mathrm{Re}(u) > \lambda$; and case (iii) in $\mathrm{Re}(u) < 0$. These three domains are shown in Fig. 1a. This definition ensures that $|\kappa(u)|$ is continuous.

This function $\kappa(u)$ is piecewise analytic. Each of the three domains is a vertical strip (or half-plane). Within each domain $\kappa(u)$ is analytic, nonzero, periodic, or antiperiodic of period πi; and can be analytically continued across the domain boundary. It satisfies the inversion relations (2.40). As $\mathrm{Re}(u) \to \pm\infty$, we see from (2.36) that $\kappa(u)$ grows as $\exp(\pm u)$.

Evaluation of $\kappa(u)$ from the Inversion Relations. Conversely, these properties are actually sufficient to define $\kappa(u)$. Because $\kappa(u)$ is analytic, nonzero, and (anti-)periodic, its logarithm must have a generalized Fourier expansion:

$$\ln \kappa(u) = Lu + \sum_{n=-\infty}^{\infty} c_n e^{2nu} \tag{4.2}$$

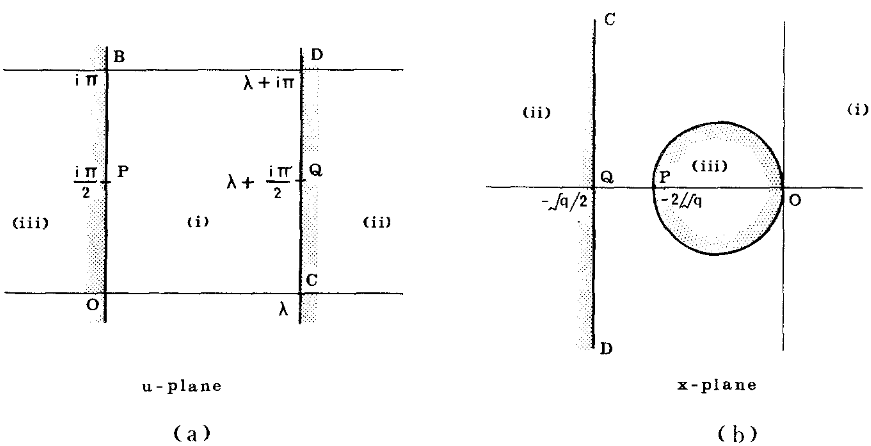

Fig. 1. The zero-field six-vertex model with $q > 4$, i.e., $|\Delta| > 1$. Domains of analyticity of κ as a function of u, and of x [Eqs. (4.1)–(4.8)].

where L is an integer. This expansion must be convergent inside the appropriate domain, and slightly beyond it.

The inversion points $u = 0$ and $u = \lambda$ both lie on the boundary of domain (i), so $\kappa(u)$ therein must satisfy both (2.40a) and (2.40b). Taking logarithms of these equations and substituting the expansion (4.2), we can solve for the coefficients L, $c_0, c_{\pm 1}, \ldots,$ thereby obtaining the result (4.1a).

For domain (ii), only the inversion point $u = \lambda$ lies on the boundary, so we can only use (2.40b). However, from the behavior when $\mathrm{Re}(u) \to \infty$ we see that $L = 1$ and $c_n = 0$ for $n \geqslant 1$. The relation (2.40b) is then sufficient to determine $c_0, c_{-1}, \ldots,$ and hence to verify the result (4.1b) for $\kappa(u)$.

Similarly, the result for domain (iii) can be obtained from the relation (2.40a) appropriate to the inversion point $u = 0$, together with the known behavior when $\mathrm{Re}(u) \to -\infty$.

The known results for the zero-field six-vertex model can thus be rederived by this simple method. To make this method rigorous, we should of course properly establish the analyticity properties that we use. Progress has been made in this direction by Shankar.[7]

At first sight we do not seem to have used the star–triangle relation (2.6) or (2.49). However, Shankar's work makes it clear that this plays a vital role in establishing that κ can be analytically continued through an inversion point.

Zeros of $Z(u)$. A useful check on (4.1) is to note from (2.1) and (2.36) that $Z(u)$ is of the form $e^{-Nu}\{d_0 + d_1 e^{2u} + d_2 e^{4u} + \cdots + d_N e^{2Nu}\}$, where $d_0, \ldots, d_N$ are constants. It can therefore be written in the form

$$Z(u) = C \prod_{j=1}^{N} \sinh(u - u_j) \tag{4.3}$$

where C is a constant and $u_1, \ldots, u_N$ are the zeros of $Z(u)$.

Take N to be large, but finite. From (2.4) and (4.1), $Z(u)$ is nonzero inside the domains (i), (ii), (iii). The zeros $u_1, \ldots, u_N$ can therefore lie only on the boundaries, i.e., the vertical lines $\mathrm{Re}(u) = 0$ and λ. They can be chosen to lie on the line segments OB and CD, and it seems likely that in the limit of N large they form a dense distribution thereon.

On OB, set $u = iy$ and let $Ng(y)\,dy$ be the number of zeros between y and $y + dy$. Taking the logarithm of (4.3), using (2.4) and the rotation symmetry $Z(u) = Z(\lambda - u)$, it follows that

$$\ln \kappa(u) = \gamma + \int_0^{\pi} \ln\left[\sinh(u - iy)\sinh(\lambda - u - iy)\right] g(y)\,dy \tag{4.4}$$

for all values of u, where γ is a constant.

We can evaluate $g(y)$ by differentiating (4.4), letting $\mathrm{Re}(u) \to 0^+$ and 0^-, and taking the difference between these limits. This gives

$$g(-iu) = (2\pi)^{-1} \frac{d}{du} \ln \left[\kappa_{\mathrm{i}}(u) / \kappa_{\mathrm{iii}}(u) \right] \tag{4.5}$$

where $\kappa_{\mathrm{i}}(u)$ is defined by (4.1a), $\kappa_{\mathrm{iii}}(u)$ by (4.1c). Thus

$$g(y) = (2\pi)^{-1} \left(1 + 2 \sum_{n=1}^{\infty} \frac{e^{-n\lambda} \cos 2ny}{\cosh n\lambda} \right) \tag{4.6}$$

for $0 < y < \pi$. This function $g(y)$ is positive, as it must be, and

$$\int_0^{\pi} g(y)\, dy = \frac{1}{2} \tag{4.7}$$

in agreement with the fact that $N/2$ of the zeros $u_1, \ldots, u_N$ must be on OB, the other $N/2$ on CD.

We can evaluate γ by using the limiting form of $\kappa(u)$ for $|\mathrm{Re}(u)|$ large, and hence write (4.4) as

$$\ln \kappa(u) = \int_0^{\pi} \ln \left[\frac{\rho_0^2 e^{\lambda} \sinh(u - iy) \sinh(u + iy - \lambda)}{\sinh^2 \lambda} \right] g(y)\, dy \tag{4.8}$$

The three results (4.1) then all follow from this single expression (4.8).

κ as a function of x. For the purpose of comparing between the $q > 4$ and $q < 4$ cases, it can be useful to change from u to the original variable x. Figure 1a then translates to 1b: domains (i) and (ii) are separated by the straight line DQC, i.e., $\mathrm{Re}(x) = -q^{1/2}/2$; (i) and (iii) are separated by the circle OPB, i.e., $\mathrm{Re}(x^{-1}) = -q^{1/2}/2$. Regarding ρ as a constant, it is evident from (2.1), (2.2), and (2.16) that Z is a polynomial in x of degree N. When N becomes large, half its zeros lie on DQC, the other half on OPB.

The function $\kappa(x)$ satisfies the inversion relations (2.32) and (2.33) in the neighborhood of the inversion points $x = 0$ and $x = \infty$, respectively. From (2.34) the right-hand side of (4.1b) is (for constant ρ) a single-valued function of x: since this function satisfies (2.33), it must do so for all values of x. In particular, it must satisfy (2.33) in the neighborhood of the *virtual* inversion point $x = -q^{1/2}/2$, i.e., the point Q in Fig. 1b. For case (ii) we could therefore have evaluated κ by the matrix inversion trick, using the point Q instead of $x = \infty$.

Similar considerations apply to case (iii) [use (2.32) near P instead of 0], and to (i) (use P, Q instead of 0, ∞), except that in case (i) the right-hand side of (4.1a) is a two-valued function of x. However, the values differ only in sign, and the only resulting modification is to negate the right-hand side of (2.32) and (2.33).

4.2. SDP Model

All the previous remarks of this section apply to the six-vertex model. The critical hard-hexagon model has $q < 4$, so does not fall into this regime. Provided boundary conditions do not matter, the Temperley–Lieb equivalence says that the SDP model should have the same value of κ as the six-vertex model. This certainly appears to be true (from large-q expansions) for the physical case, when $0 < x < \infty$ and $0 < u < \lambda$. More generally, it appears to be true throughout domain (i) in Fig. 1. It is not clear that it is true inside domains (ii) and (iii): indeed there are objections to applying the frozen KDP six-vertex states to the Potts model. They correspond to all arrows pointing the same way, say generally upward. When one obtains the Temperley–Lieb equivalence graphically for a finite lattice, one uses boundary conditions that prohibit this arrow configuration.[19] Looked at in a rather more general way, from (2.1), (2.2), (2.36), and (4.1c) it is apparent that the case (iii) solution corresponds to using a subspace in which $U_1 = \cdots = U_n = 0$. The six-vertex representation (3.12) has such a trivial subspace, but the Potts model one in (3.31) does not. In any case, such a subspace cannot contribute to Z_f, as defined by (2.26) and (2.24), though it can and does contribute to the six-vertex partition function when cyclic boundary conditions are used.

It seems that boundary conditions are significant in cases (ii) and (iii). Admittedly the SDP model is then unphysical in that it has nonpositive Boltzmann weights, but it would still be interesting to obtain κ as a function of u throughout the whole complex plane. I have not been able to do this, but in the next section I shall show that a similar problem arises for $0 < q < 4$. One then has an excellent guide, namely, the $q = 2$ Ising case, and can see how to evaluate κ in a plausible manner by appropriate use of the matrix inversion technique, provided that $1 < q < 4$.

5. THE FUNCTION κ FOR $q < 4$

5.1. Six-Vertex Model

For $q < 4$ we use the parameters μ, ρ_0, v, which are related to q, ρ, and x by (2.41). They are also related to the six-vertex weights a, b, c by (3.15), and from (3.13) the condition $0 < q < 4$ implies that $|\Delta| < 1$. In this case the six-vertex model corresponds to a critical eight-vertex model. If a, b, c are real, then we can choose μ, v to be real, satisfying (2.42), and the six-vertex results[23] are as follows.

(i) $0 < v < \mu$:

$$\ln \kappa = \ln \rho_0 + \int_{-\infty}^{\infty} \frac{\cosh(\pi - 2\mu)t \sinh vt \sinh(\mu - v)t}{t \sinh \pi t \cosh \mu t} \, dt \qquad (5.1a)$$

(ii) $\mu < v < \pi$:

$$\ln \kappa = \ln \rho_0 + \int_{-\infty}^{\infty} \frac{\cosh(\pi - 2\mu)t \sinh(\pi - v)t \sinh(v - \mu)t}{t \sinh \pi t \cosh(\pi - \mu)t} dt \quad (5.1b)$$

These equations are true for v real. Guided by the $q > 4$ regime, and by the requirement that $|\kappa(v)|$ be continuous, it seems reasonable to suppose that they are also true for complex values of v, provided that case (i) is taken to apply in the domain $0 < \mathrm{Re}(v) < \mu$; and case (ii) in $\mu < \mathrm{Re}(v) < \pi$. These two domains are shown in Fig. 2a, and the corresponding domains in the complex x plane are shown in Fig. 2b. As for $q > 4$, we see that the domain boundaries lie on the lines $\mathrm{Re}(x^{\pm 1}) = -q^{1/2}/2$, but now they occupy only part of these lines.

This function $\ln \kappa(v)$ is analytic in each of the vertical strips (i) and (ii) in Fig. 1a, and can be analytically continued across the boundaries. It satisfies the inversion relations (2.47), and from (2.43) we see that as $\mathrm{Im}(v) \to \pm\infty$, $\kappa(v)$ grows as $\exp(\mp iv)$.

These properties actually define $\kappa(v)$, just as the corresponding properties for $q > 4$ define $\kappa(u)$. The analyticity and growth rate properties imply that $d^2 \ln \kappa(v)/dv^2$ is Fourier integrable inside each domain, i.e.,

$$\frac{d^2 \ln \kappa(v)}{dv^2} = \int_{-\infty}^{\infty} c(t) e^{2vt} dt \quad (5.2)$$

where $c(t)$ is analytic in some strip about the real t axis, and the integral converges for v inside the domain, and just beyond it.

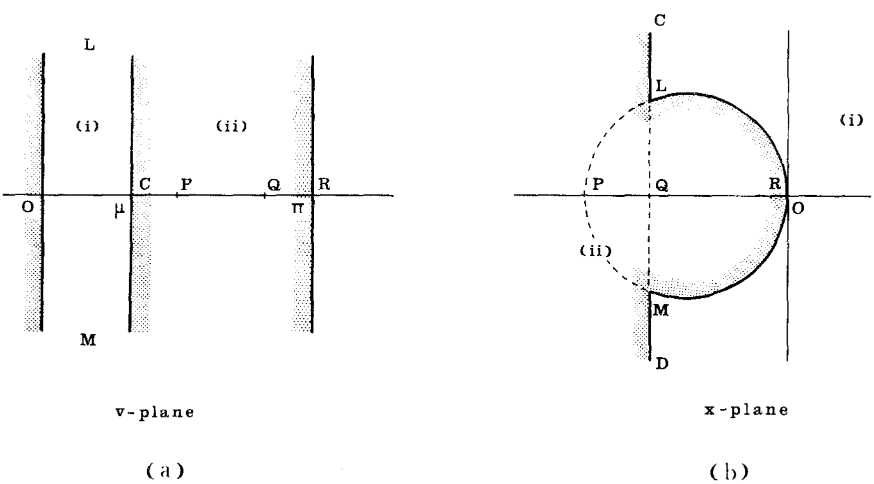

Fig. 2. The zero-field six-vertex model with $0 < q < 4$, i.e., $|\Delta| < 1$. Domains of analyticity of κ as a function of v, and of x [Eqs. (5.1)–(5.8)].

The inversion points $v = 0$ and $v = \mu$ both lie on the boundary of domain (i), so $\kappa(v)$ therein must satisfy both (2.47a) and (2.47b). Taking logarithms of these equations, differentiating twice, substituting the form (5.2) and inverting the Fourier integral, we obtain the equations

$$\begin{aligned} c(t) + c(-t) &= -4t\cosh(\pi - 2\mu)t/\sinh \pi t \\ c(t) + e^{-4\mu t}c(-t) &= -4te^{-2\mu t}\cosh(\pi - 2\mu)t/\sinh \pi t \end{aligned} \tag{5.3}$$

These can immediately be solved for $c(t)$. Substituting the result into (5.2), integrating twice [the constants of integration can themselves be obtained from (2.47)] and rearranging, we obtain (5.1a).

For case (ii), we note from (2.43) that $X_j(v - \pi) = -X_j(v)$, so v on the left-hand side of (2.50) can be replaced by $v - \pi$. This means that $v = \pi$ is also an inversion point (X_j is there proportional to the identity matrix) and that

$$\kappa(v - \pi)\kappa(\pi - v) = \rho_0^2 \sin(\mu - v)\sin(\mu + v)/\sin^2\mu \tag{5.4}$$

The inversion points $v = \mu$ and $v = \pi$ lie on the boundary of domain (ii), so we can use the corresponding equations (2.47b) and (5.4). Substituting (5.2) and proceeding similarly to case (i), we obtain the result (5.1b).

As for the $q > 4$ regime, it is helpful to look at the distribution of the zeros of the partition function. The equation (4.3) now becomes

$$Z(v) = C\prod_{j=1}^{N} \sin(v - v_j) \tag{5.5}$$

where we can choose each v_j so that $0 \leqslant \mathrm{Re}(v_j) < \pi$. The results (5.1), extended to the domains (i) and (ii) of Fig. 2, imply for N large that $v_1, \ldots, v_N$ lie on the lines $\mathrm{Re}(v) = 0$ and $\mathrm{Re}(v) = \mu$. On the imaginary axis, set $v = iy$ and let $Ng(y)\,dy$ be the number of zeros between y and $y + dy$. Proceeding as in (4.4)–(4.8), we obtain

$$g(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos 2yt \cosh(\pi - 2\mu)t}{\cosh \mu t \cosh(\pi - \mu)t}\,dt \tag{5.6}$$

$$\int_{-\infty}^{\infty} g(y)\,dy = \frac{1}{2} \tag{5.7}$$

$$\ln\kappa(v) = \int_{-\infty}^{\infty} \ln\left[\frac{\rho_0^2\sin(v - iy)\sin(\mu - v - iy)}{\sin^2\mu}\right] g(y)\,dy \tag{5.8}$$

Equation (5.7) implies that $N/2$ of the zeros lie on the imaginary v axis, the other $N/2$ on the vertical line $\mathrm{Re}(v) = \mu$.

Regarding κ as a function of x (holding ρ fixed), we obtain the picture in Fig. 2b. The domains (i) and (ii) are separated by the path $CLOMD$, made up of segments of the straight line $\mathrm{Re}(x) = -q^{1/2}/2$, and the circle $\mathrm{Re}(x^{-1}) = -q^{1/2}/2$. The virtual inversion points $x = -2q^{-1/2}$ and $-q^{1/2}/2$ (points P and Q in the figure) lie inside the domain (i), and the inversion relations (2.32) and (2.33) are *not* satisfied in their vicinity.

5.2. SDP Model for $1 < q < 4$

The self-dual Potts model has, for arbitrary q, been solved only by using the Temperley–Lieb equivalence to a six-vertex model. An important exception is the Ising model case $q = 2$, which has been solved[14,28] for all values of K and L in (3.28). Regard R therein as a constant, and take $\mathfrak{L}$ to be the square lattice with $N/2$ sites, with cyclic (i.e., toroidal) boundary conditions, and an even number of rows and columns. By reversing spins on alternate rows (or columns), and noting that there must be an even number of unlike spin pairs in each column (or row), one can readily establish for $q = 2$ that

$$Z(e^K, e^L) = e^{NK/2} Z(e^{-K}, -e^L) = e^{NL/2} Z(-e^K, e^{-L}) \tag{5.9}$$

These symmetry relations do not affect the self-duality condition (3.32), so we can check whether they are satisfied by the six-vertex solution (5.1). They are not, so we cannot use the Temperley–Lieb equivalence for all values of x or v.

Instead we can of course directly use Onsager's result[14,28] for the Ising model:

$$\ln \kappa = \ln(2^{1/2} R) + \frac{1}{4}(K + L) + \frac{1}{4(2\pi)^2} \int\int_{-\pi}^{\pi} \ln(\cosh K \cosh L - \sinh K \cos\alpha - \sinh L \cos\beta)\, d\alpha\, d\beta \tag{5.10}$$

This is true (with appropriate choices of the branch of the logarithm function) for all values of K and L, real or complex. When the self-duality relation (3.32) (which for $1 \leqslant q \leqslant 4$ is the criticality condition) is satisfied, then, using (3.33) and (2.41), we find that there are four cases to consider, corresponding to the four domains in Fig. 3. [The boundary lines *OLPM* and *CLMD* correspond to $\sinh K$ and $\sinh L$ being pure imaginary, which is when the argument of the logarithm in (5.2) vanishes for real nonzero values of α and β.]

In domain (i) we find that (5.10) is indeed the same as (5.1a), with $\mu = \pi/4$, but in the other domains it is not the same as the corresponding six-vertex result (5.1b). The Temperley–Lieb equivalence therefore definitely fails in these domains for $q = 2$, and presumably for all $q < 4$, just as

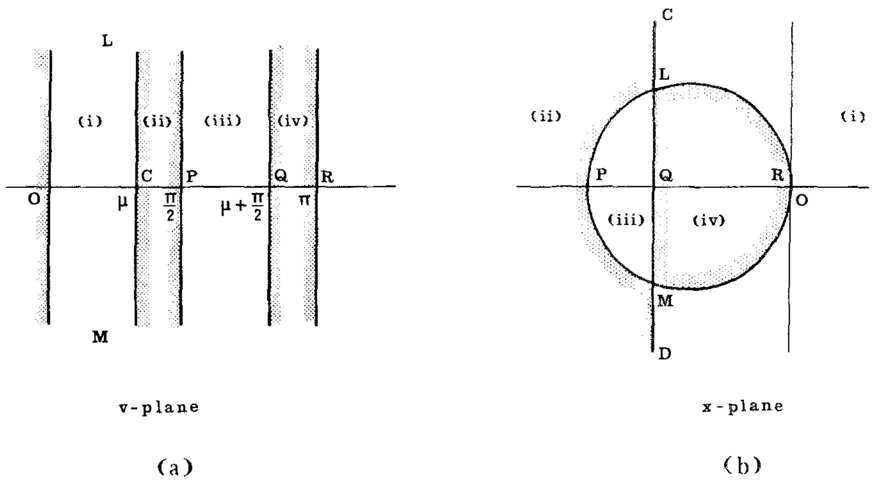

Fig. 3. The self-dual Potts (SDP) model with $0 < q < 4$. Domains of analyticity of κ as a function of v, and of x [Eqs. (5.12)–(5.15)].

it appears to fail for $q > 4$. These failures must be due to the six-vertex model being sensitive to boundary conditions.

Of course, domains (ii)–(iv) are unphysical, in that the Potts model Boltzmann weights are not all positive. Even so, we need to consider such cases to obtain a full understanding of κ as a function of x or v. Further, we shall find that they correspond to physical cases of the hard-hexagon model.

Fortunately the $q = 2$ case also suggests how to obtain $\ln \kappa$ for the SDP model for other values of q: its solution is analytic within each of the domains (i)–(iv) and can be analytically continued across the domain boundaries; the inversion relations (2.32) and (2.33) are satisfied not only at the points $x = 0$ and $x = \infty$, but also at the virtual inversion points P and Q, respectively. In terms of the variable v in (2.41), this means that we have the additional inversion relations

$$\kappa(v)\kappa_{ac}(\pi - v) = -\rho_0^2 \sin(\mu - v)\sin(\mu + v)/\sin^2\mu \qquad (5.11a)$$

$$\kappa(v)\kappa_{ac}(\pi + 2\mu - v) = -\rho_0^2 \sin v \sin(2\mu - v)/\sin^2\mu \qquad (5.11b)$$

Just as (2.47a) corresponds to (2.50), so does (5.11a) correspond to (2.50) after using the fact that $X_j(v)$ is antiperiodic of period π. Equation (5.11b) follows from (5.11a) and the rotation symmetry (2.46). It therefore seems reasonable to suppose that they are true for all real values of q within some neighborhood of $q = 2$. We can then use these, together with analyticity and the original inversion relations (2.47) and (5.4), to obtain $\kappa(v)$.

The calculation proceeds in the way outlined after (5.2), only now we can use the inversion points $v = 0$ and μ for domain (i), $v = \mu$ and $\frac{1}{2}\pi$ for (ii), $v = \frac{1}{2}\pi$ and $\mu + \frac{1}{2}\pi$ for (iii), and $v = \mu + \frac{1}{2}\pi$ and π for (iv). Thus in (i) we use the equations (2.47a) and (2.47b), in (ii) we use (2.47b) and (5.11a), in (iii) we use (5.11a) and (5.11b), and in (iv) we use (5.11b) and (5.4). The results are as follows:

(i) $0 < \mathrm{Re}(v) < \mu$:

$$\ln \kappa = \ln \rho_0 + \int_{-\infty}^{\infty} \frac{\cosh(\pi - 2\mu)t \sinh vt \sinh(\mu - v)t}{t \sinh \pi t \cosh \mu t} dt \tag{5.12a}$$

(ii) $\mu < \mathrm{Re}(v) < \frac{1}{2}\pi$:

$$\ln \kappa = \ln \rho_0 + \int_{-\infty}^{\infty} \frac{\sinh(v - \mu)t\, \phi(\mu, v, t)}{t \sinh \pi t \sinh(\pi - 2\mu)t} dt \tag{5.12b}$$

where

$$\begin{aligned}\phi(\mu, v, t) = {} & \sinh(\pi - \mu)t \sinh(\pi - 2\mu - v)t \\ & + \sinh \mu t \sinh(v - 2\mu)t\end{aligned} \tag{5.12c}$$

(iii) $\frac{1}{2}\pi < \mathrm{Re}(v) < \mu + \frac{1}{2}\pi$:

$$\ln \kappa = \ln \rho_0 + \int_{-\infty}^{\infty} \frac{\psi(\mu, v, t)}{2t \sinh \pi t \cosh \mu t} dt \tag{5.12d}$$

where

$$\psi(\mu, v, t) = \cosh \mu t \cosh(\pi - 2\mu)t - \cosh 2\mu t \cosh(2v - \pi - \mu)t \tag{5.12e}$$

(iv) $\mu + \frac{1}{2}\pi < \mathrm{Re}(v) < \pi$:

$$\ln \kappa = \ln \rho_0 + \int_{-\infty}^{\infty} \frac{\sinh(\pi - v)t\, \phi(\mu, \pi + \mu - v, t)}{t \sinh \pi t \sinh(\pi - 2\mu)t} dt \tag{5.12f}$$

The results for any other value of v can be obtained from the periodicity property $\kappa(v) = \kappa(v + \pi)$. One can easily see that the symmetry relation (2.46) is satisfied, and can verify that $|\kappa(v)|$ is continuous across the domain boundaries. There is some ambiguity in the choice of signs in (5.11), corresponding to the fact that $X_j(v)$ is antiperiodic in v, while we would normally choose the logarithm in (2.4) to ensure that $\kappa(v)$ is periodic. I have chosen the signs to ensure that $\kappa(v)$ is positive real for real values of v.

The zeros $v_1, \ldots, v_N$ in (5.5) now lie on the lines $\mathrm{Re}(v) = 0$, μ, $\frac{1}{2}\pi$, and $\mu + \frac{1}{2}\pi$. As before, let $Ng(y)dy$ be the number of zeros on the imaginary axis, between iy and $i(y + dy)$. Similarly, let $Nh(y)dy$ be the number on the vertical line $\mathrm{Re}(v) = \frac{1}{2}\pi$, between $\frac{1}{2}\pi + iy$ and

$\frac{1}{2}\pi + i(y + dy)$. Then from (5.12) it follows that

$$
\begin{aligned}
g(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos 2yt \sinh(\pi - 3\mu)t}{\cosh \mu t \sinh(\pi - 2\mu)t}\, dt \\
h(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\cos 2yt \sinh \mu t}{\cosh \mu t \sinh(\pi - 2\mu)t}\, dt
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
\ln \kappa(v) = &\int_{-\infty}^{\infty} \ln\left[\frac{\rho_0^2 \sin(v - iy)\sin(\mu - v - iy)}{\sin^2\mu}\right] g(y)\, dy \\
&+ \int_{-\infty}^{\infty} \ln\left[\frac{\rho_0^2 \sin(v - \frac{1}{2}\pi - iy)\sin(\mu - v + \frac{1}{2}\pi - iy)}{\sin^2\mu}\right] h(y)\, dy
\end{aligned}
\tag{5.14}
$$

this last equation being true throughout the complex v plane.

From (5.13),

$$
\begin{aligned}
\int_{-\infty}^{\infty} g(y)\, dy &= \tfrac{1}{2}(\pi - 3\mu)/(\pi - 2\mu) \\
\int_{-\infty}^{\infty} h(y)\, dy &= \tfrac{1}{2}\, \mu/(\pi - 2\mu)
\end{aligned}
\tag{5.15}
$$

so there are $\frac{1}{2}N(\pi - 3\mu)/(\pi - 2\mu)$ zeros on the imaginary v axis, and on the line $\mathrm{Re}(v) = \mu$. There are also $\frac{1}{2}N\mu/(\pi - 2\mu)$ zeros on the line $\mathrm{Re}(v) = \frac{1}{2}\pi$, and on $\mathrm{Re}(v) = \mu + \frac{1}{2}\pi$. Altogether there are N zeros, as is required by (5.5). In the x plane the first two sets of zeros lie on the lines LOM and $LCDM$, while the second two sets lie on LPM and LQM.

For the Ising case we have $q = 2$ and $\mu = \pi/4$. There are then $N/4$ zeros on each line, and (5.14) is indeed equivalent to the critical case of (5.10). By using the matrix inversion method, we have generalized this result to other values of q less than 4.

This generalization makes perfectly good sense for $1 < q < 4$, since then $\pi/3 > \mu > 0$ and $g(y)$ and $h(y)$ are positive, as they must be. For $q = 4$ ($\mu = 0$), all the zeros lie on the lines $\mathrm{Re}(v) = 0$, μ. As q decreases, the number of zeros on these lines decreases, while zeros appear instead on the lines $\mathrm{Re}(v) = \frac{1}{2}\pi$ and $\mu + \frac{1}{2}\pi$. This process continues as q approaches 1, until all the zeros lie on the latter pair of lines.

5.3. SDP Model for $0 < q \leqslant 1$

The distribution functions $g(y)$ and $h(y)$ must be positive, so the results (5.12)–(5.15) cannot be true for $0 < q < 1$, i.e., $\pi/2 > \mu > \pi/3$. For

$q = 1$, i.e., $\mu = \pi/3$, the Potts model is trivial. All spins have value 1, and from (3.28), using (3.31) and (2.41),

$$\kappa = \mathrm{R}e^{(K+L)/2} = \rho(1 + x) = \rho_0 \sin(\mu + v)/\sin \mu \tag{5.16}$$

for all complex values of x and v.

The equations (5.12) correctly give this result in domains (i), (ii), and (iv), but are incorrect in (iii). Indeed they must fail somewhere, since they imply that Z has zeros on the contour $LPMQL$ in Fig. 3b, while from (5.16) all its zeros coalesce at $x = -1$, which point lies inside the contour. Thus in domain (iii) the limits $N \to \infty$ and $q \to 1$ cannot be interchanged.

For $0 < q < 1$, I have no reasonable suggestions to make for κ. The most obvious idea seems to be to ignore the very special $q = 1$ case, and to suppose that $v_1, \ldots, v_N$ "stick" on the lines $\mathrm{Re}(v) = \frac{1}{2}\pi$ and $\mu + \frac{1}{2}\pi$, there being no zeros on $\mathrm{Re}(v) = 0$ and μ. This implies that $\ln \kappa$ is analytic for $\mu - \frac{1}{2}\pi < \mathrm{Re}(v) < \frac{1}{2}\pi$, which domain is the union of (i), (ii), and (iv) (after allowing for periodicity of period π in the v plane). The obvious thing to do then is to calculate κ from the virtual inversion points $v = \mu - \frac{1}{2}\pi$ and $\frac{1}{2}\pi$, but this implies that the left-hand side of (5.11a) has no zeros in the combined domain, while the right-hand side clearly vanishes at $v = \mu$.

Another idea is to maintain that (5.12a) should still be valid, but should extend throughout the combined domain, while (5.12d) should still be valid inside (iii). Unfortunately this means that $|\kappa(v)|$ is discontinuous at $\mathrm{Re}(v) = \frac{1}{2}\pi$ and $\mu + \frac{1}{2}\pi$. Still another idea is to apply the six-vertex result (5.1) throughout the complex plane, but this implies that the zeros all jump back onto the lines $\mathrm{Re}(v) = 0$ and μ, which seems unreasonable.

The SDP model for $0 < q < 1$ thus remains unsolved, at least outside domain (i). This includes the real antiferromagnetic case. Wu[10] [in his Eq. (5.28)] has remarked that this case presents a problem in that it is not a transition point, whereas the corresponding six-vertex model (obtained by using the Temperley–Lieb equivalence) is critical.

5.4. Critical Hard-Hexagon Model

This model is defined in (3.17)–(3.23). It has been solved exactly by the matrix inversion relations,[3] and recently by more conventional transfer matrix methods.[8] Define

$$r(v) = \omega_4\omega_5/\omega_1 = \rho_0 \frac{\sin(\mu + v)\sin(2\mu - v)}{\sin \mu \sin(2\mu + v)} \tag{5.17}$$

then the solution is

(i) $0 < \mathrm{Re}(v) < \mu$:

$$\kappa = r(v) \tag{5.18a}$$

(ii) $\mu < \mathrm{Re}(v) < \frac{1}{2}\pi$:

$$\kappa = r(v)\sin(5v/3)/\sin\left[5(2\mu - v)/3\right] \tag{5.18b}$$

(iii) $\frac{1}{2}\pi < \mathrm{Re}(v) < \mu + \frac{1}{2}\pi$:

$$\kappa = -r(v)\cot(5v/2) \tag{5.18c}$$

(iv) $\mu + \frac{1}{2}\pi < \mathrm{Re}(v) < \pi$:

$$\kappa = r(v)\sin(5v/3)/\sin\left[5(2\mu + v)/3\right] \tag{5.18d}$$

[Equation (5.18a) corresponds to (43b) of Ref. 3, u therein being replaced by our v; (5.18b) and (5.18d) correspond to (43a); (5.18c) to the critical case of Eq. (6.28c) of the next section.]

Note from (3.23) that for this model $\mu = \pi/5$. If we substitute this value into (5.12), then we obtain exactly the equations (5.18). Thus the critical hard-hexagon model is equivalent to the corresponding six-vertex model only in domain (i), but is always equivalent to the self-dual Potts model with q given by (3.25), i.e., $q = \frac{1}{2}(3 + \sqrt{5}) = 2.618\ldots$.

The proper hard-hexagon model (i.e., the triangular lattice gas with nearest-neighbor exclusion) is obtained by setting $v = 2\mu$, so lies in domain (ii). This is therefore equivalent [in the weak sense of having the same κ, and X_j's satisfying (2.16)–(2.19)] to the Potts model, but not to the six-vertex model.

6. MODELS INVOLVING ELLIPTIC FUNCTIONS

The zero-field eight-vertex model, and the noncritical hard-hexagon model, also have local transfer matrices which permit nontrivial solutions of the star–triangle relation (2.6). However, the solutions no longer have the linearity property (2.10), and I know of no analog of the algebraic relations (2.16) and (2.19).

Even so, the star–triangle relation can still be written more explicitly as (2.49), the symmetry relation (2.39) is valid, and the inversion relation (2.50) has a straightforward generalization. They can therefore be used[6,7] to calculate κ, as I shall indicate here. It turns out that there are two possibilities, just as in Section 5 we found the results (5.1) and (5.12) for the six-vertex and SDP models, respectively.

6.1. Eight-Vertex Model

This model is defined in Section 3.1. Without loss of generality we can take $s = t = 1$ in (3.6), leaving four independent parameters a, b, c, d. We can introduce four related parameters $w_1, \ldots, w_4$ by

$$\begin{aligned} w_1 &= a - b, \qquad & w_2 &= a + b \\ w_3 &= c + d, \qquad & w_4 &= c - d \end{aligned} \tag{6.1}$$

The partition function per site satisfies (3.7), (3.8), and other symmetry and duality properties.[12] These are particularly simple if we think of κ as a function of $w_1, \ldots, w_4$, for then they can all be combined into the single equation[1]

$$\kappa(w_1, w_2, w_3, w_4) = \kappa(\pm w_i, \pm w_j, \pm w_k, \pm w_l) \tag{6.2}$$

for independent choices of the $\pm$ signs, and for all permutations (i, j, k, l) of $(1,2,3,4)$. Thus κ is unaltered by negating or interchanging any of the w's.

Let $\theta_1(u,q), \ldots, \theta_4(u,q)$ be the usual elliptic theta functions of half-period π:

$$\begin{aligned} \theta_1(u,q) &= 2q^{1/4}\sin u \prod_{n=1}^{\infty} (1 - q^{2n}e^{2iu})(1 - q^{2n}e^{-2iu})(1 - q^{2n}) \\ \theta_2(u,q) &= \theta_1(u + \tfrac{1}{2}\pi, q) \\ \theta_4(u,q) &= \prod_{n=1}^{\infty} (1 - q^{2n-1}e^{2iu})(1 - q^{2n-1}e^{-2iu})(1 - q^{2n}) \\ \theta_3(u,q) &= \theta_4(u + \tfrac{1}{2}\pi, q) \end{aligned} \tag{6.3}$$

Here u is known as the "argument" of these functions, q as the "nome," $|q| < 1$.

Notation. In this section q (without a suffix) will always be an elliptic nome. We will regain contact with Sections 2–5 in the limits $q \to 0$ and $q^2 \to \pm 1$ (when the elliptic functions reduce to trigonometric functions): I shall hereinafter use the symbol q_P to denote the q of those previous sections.

It is also useful to define

$$\phi_1(u,q) = -i\theta_1(iu,q), \qquad \phi_j(u,q) = \theta_j(iu,q), \qquad j = 2,3,4 \tag{6.4}$$

These functions are real, ϕ_j bearing a similar relationship to θ_j as $\sinh u$ does to $\sin u$.

We can now define four parameters ρ_0, q, λ, u by

$$w_j = \rho_0 \frac{\phi_j(\tfrac{1}{2}\lambda - u, q)}{\phi_j(\tfrac{1}{2}\lambda, q)}, \qquad j = 1, \ldots, 4 \tag{6.5}$$

and can regard κ as a function of ρ_0, q, λ, u. More particularly, let us regard ρ_0, q, λ as fixed and u as a variable. [When $q = 0$, then $d = 0$ and we regain the six-vertex parametrization (3.14).]

Replacing u by $\lambda - u$ merely negates w_1, so we regain the rotation symmetry (2.39). We can also verify that (2.49) and (2.50) are satisfied, provided only that the function $\sinh u$ therein is replaced by $\phi_1(u,q)$. Thus we also expect the inversion relations (2.40) to be valid, subject to this substitution, i.e.,

$$\kappa(u)\kappa_{ac}(-u) = \rho_0^2\phi_1(\lambda - u)\phi_1(\lambda + u)/\phi_1^2(\lambda) \tag{6.6a}$$

$$\kappa(u)\kappa_{ac}(2\lambda - u) = \rho_0^2\phi_1(u)\phi_1(2\lambda - u)/\phi_1^2(\lambda) \tag{6.6b}$$

writing $\phi_j(u,q)$ simply as $\phi_j(u)$.

Incrementing u in (6.5) by $i\pi$ leaves w_3 and w_4 unchanged, while negating w_1 and w_2. From (6.2), this leaves κ unchanged. Incrementing u by $\ln q$ multiplies each w_j by $\pm q^{-1}e^{2u-\lambda}$. Thus κ satisfies the two quasiperiodicity relations

$$\kappa(u) = \kappa(u + i\pi) \tag{6.7a}$$

$$= qe^{\lambda - 2u}\kappa(u - \ln q) \tag{6.7b}$$

Incrementing λ by $i\pi$ (or $\ln q$) rearranges $w_1, \ldots, w_4$ and multiplies them by factors which differ only in sign, so there are similar periodicity relations for κ as a function of λ. We can therefore, without loss of generality, restrict u and λ to be complex numbers with real parts between 0 and $\ln|1/q|$. Define a function $S(w, y, p)$ by

$$S(w, y, p) = \sum_{n=1}^{\infty} \frac{(1 - w^n)(1 - y^n w^{-n})(y^n + y^{-n}p^n)}{n(1 - p^n)(1 + y^n)} \tag{6.8}$$

for $|p| < |y| < |w| < 1$. Then the solution[17] of the eight-vertex model is

(i) $0 < \mathrm{Re}(u) < \mathrm{Re}(\lambda) < \ln|1/q|$:

$$\ln\kappa = \ln\rho_0 + S(e^{-2u}, e^{-2\lambda}, q^2) \tag{6.9a}$$

(ii) $0 < \mathrm{Re}(\lambda) < \mathrm{Re}(u) < \ln|1/q|$:

$$\ln\kappa = \ln\rho_0 + u - \lambda + S(e^{2\lambda - 2u}, q^2e^{2\lambda}, q^2) \tag{6.9b}$$

The solution in case (i) satisfies the two inversion relations (6.6), and can be obtained from them by using the properties that $\ln\kappa(u)$ is analytic in the vertical strip $0 < \mathrm{Re}(u) < \mathrm{Re}(\lambda)$, and is periodic of period πi. [These properties imply that $\ln\kappa$ has a Fourier series in integer powers of e^{2u}: the coefficients can then be obtained[6,7] from (6.6).]

Similarly, the solution in case (ii) is given by (6.6b) and

$$\kappa(u)\kappa_{ac}(2\tau - u) = \rho_0^2 q^{-2}e^{-2\lambda}\phi_1(\tau + \lambda - u)\phi_1(\lambda + u - \tau)/\phi_1^2(\lambda) \tag{6.10}$$

where

$$q = e^{-\tau} \tag{6.11}$$

This inversion relation (6.10) is a consequence of (6.6a) and (6.7).

The partition function Z is an entire function of u, and has the same periodicity properties (even for N finite) as $\kappa^N(u)$. From (6.7) and basic complex variable theory it follows that

$$Z(u) = C \prod_{j=1}^{N} \phi_1(u - u_j) \tag{6.12}$$

where C and $u_1, \ldots, u_N$ are constants, $0 \leqslant \mathrm{Im}(u_j) \leqslant \pi$,

$$\sum_{j=1}^{N} u_j = \tfrac{1}{2} N\lambda + im\pi \tag{6.13}$$

where m is an integer. Assuming as with (4.3) that $u_1, \ldots, u_N$ tend to dense distributions on the lines $\mathrm{Re}(u) = 0$ and $\mathrm{Re}(\lambda)$, and that $Ng(y)dy$ is the number between iy and $i(y + dy)$, we find that

$$g(y) = (2\pi)^{-1}\left[1 + 4\sum_{n=1}^{\infty} \frac{\cos 2ny(e^{-4n\lambda} + q^{2n})}{(1 + e^{-2n\lambda})(e^{-2n\lambda} + q^{2n})}\right] \tag{6.14}$$

$$\ln \kappa(u) = \int_0^{\pi} \ln\left[\frac{\rho_0^2 \phi_1(u - iy)\phi_1(u + iy - \lambda)}{\phi_1(iy)\phi_1(\lambda - iy)}\right] g(y)\, dy \tag{6.15}$$

These results satisfy the consistency condition

$$\int_0^{\pi} g(y)\, dy = \tfrac{1}{2} \tag{6.16}$$

implying that there are indeed N u_j's, half of them lying on the imaginary axis, the other half on the line $\mathrm{Re}(u - \lambda) = 0$.

From now on let us take λ to be real and positive. Then the eight-vertex results (6.9)–(6.16) are very reminiscent of the $q_p > 4$ (i.e., $|\Delta| > 1$) results (4.1)–(4.8) of the zero-field six-vertex model. In fact they reduce to them in the limit when $q \to 0$, ρ_0, u, λ being held fixed.

From (6.3)–(6.5), negating q merely interchanges w_3 and w_4, so κ must be an even function of q, as is evident from (6.9). It is really q^2, rather than q, that enters the equations.

The Limits $q^2 \to 1$ and -1. Let us take q^2 to be real, between -1 and 1, and consider the limits $q^2 \to 1$ and $q^2 \to -1$. We can handle the former case by using the "conjugate nome" identities:

$$\phi_j(u, e^{-\tau}) = \left(\frac{\pi}{\tau}\right)^{1/2} e^{u^2/\tau} \theta_{j'}\left(\frac{\pi u}{\tau}, e^{-\pi^2/\tau}\right) \tag{6.17}$$

where as j runs from 1 to 4, j' successively takes the values 1, 4, 3, 2. Set $q = e^{-\tau}$ and make these substitutions in (6.5). Let $\tau \to 0$, keeping the quantities

$$\rho_0' = \rho_0 \exp\left[-u(\lambda - u)/\tau\right]$$
$$\mu = \pi\lambda/\tau, \qquad v = \pi u/\tau \tag{6.18}$$

fixed. Then by using (6.3) we obtain

$$w_1 = \rho_0' \frac{\sin(\frac{1}{2}\mu - v)}{\sin(\frac{1}{2}\mu)}, \qquad w_4 = \rho_0' \frac{\cos(\frac{1}{2}\mu - v)}{\cos(\frac{1}{2}\mu)}$$
$$w_2 = w_3 = \rho_0' \tag{6.19}$$

Interchanging w_2 with w_4 and using (6.1), we therefore again have a six-vertex model, only now $q_p < 4$ and the weights a, b, c are given by (3.15), ρ_0 therein being replaced by ρ_0'. The results (6.9)–(6.16) reduce to the corresponding six-vertex results (5.1)–(5.8). In particular, cases (i) and (ii) in (6.9) correspond to the domains (i) and (ii) in Fig. 2a.

Now consider instead the limit $q^2 \to -1$. First we set

$$q = ie^{-\tau}, \qquad \rho_0' = \rho_0 \exp\left[-u(\lambda - u)/\tau\right]$$
$$\mu = \tfrac{1}{2}\pi\lambda/\tau, \qquad v = \tfrac{1}{2}\pi u/\tau \tag{6.20}$$

and use the identities

$$\phi_j(u, ie^{-\tau}) = \zeta_j\left(\frac{\pi}{2\tau}\right)^{1/2} e^{u^2/\tau}\theta_{j'}\left(\frac{\pi u}{2\tau}, ie^{-\pi^2/4\tau}\right) \tag{6.21}$$

where now $j' = 1,2,4,3$ for $j = 1,2,3,4$, respectively, and $\zeta_1, \ldots, \zeta_4 = 1, 1, (1+i)/\sqrt{2}, (1-i)/\sqrt{2}$. Then we let τ, λ, $u \to 0$, keeping ρ_0', μ, and v fixed. Again we obtain (6.19) (except that w_2 and w_4 are interchanged), so again we have a six-vertex model with weights given by (3.15).

However, instead of obtaining the standard six-vertex results (5.1)–(5.8), we find instead that (6.9)–(6.16) become the SDP results (5.12)–(5.15), with their four distinct domains.

We can see how this comes about from Fig. 4 and the periodicity relations (6.7). The function $\kappa(u)$ has periods $i\pi$ and $\tau + \frac{1}{2}i\pi$. This means that κ in domain (i) is repeated in (iii), but is shifted by $\frac{1}{2}\pi i$. Thus OC (or $O''C''$) corresponds to $O'C'$. Similarly, domain (ii) is repeated, but shifted, in domain (iv).

When we let $\tau \to 0$, keeping μ and v fixed, we focus on a strip about the real axis $OCPQR$, of width of order τ. Points such as O', C', P' therefore go off to infinity in the v plane, and we are left with four distinct domains (i) to (iv).

We can see what this means in terms of the zeros $u_1, \ldots, u_N$ of the partition function. Half of these lie on the line $OP'O''$, half on $CQ'C''$. As $\tau \to 0$ they cluster about the points O and P', and C and Q'. Those in the vicinity of P' and Q' disappear from direct consideration, but instead we

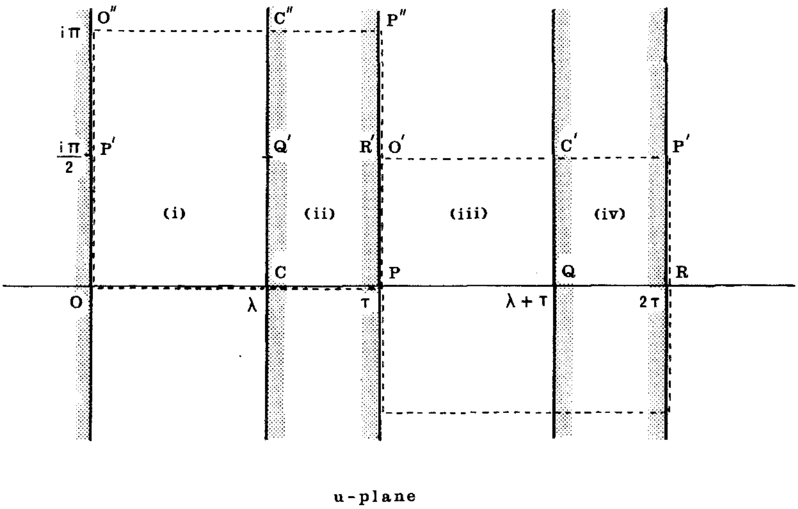

Fig. 4. The zero-field eight-vertex model with pure imaginary nome $q = ie^{-\tau}$ [Eqs. (6.7)–(6.16) and (6.20)]. The broken lines outline two equivalent period rectangles, but they differ by a vertical shift of $i\pi/2$. As a result the domains (i)–(iv) become distinct in the limit when τ, λ, $u \to 0$, their ratios remaining fixed. The SDP results (5.12) are then obtained, instead of the six-vertex results (5.1).

see their period repeats at P and Q. The function g in (5.13) gives the distribution of zeros near O and C; h gives it near P and Q.

We can also see how it comes about that $\kappa(v)$ satisfies the inversion relations (5.11) in the neighborhood of the virtual inversion points P and Q: this is just a consequence of the appropriate relations about the true inversion points O' and C', together with the fact that $\kappa(u)$ is analytic in each vertical strip, and is periodic of period πi. For instance, (6.10) applies to the domain (ii) and domain (iii) forms of κ, except that τ therein is to be replaced by $\tau + \frac{1}{2}\pi i$. Each of these functions is periodic (or antiperiodic) of period πi, so we can in turn replace $\kappa_{ac}(2\tau + i\pi - u)$ in (6.10) by $\pm\, \kappa_{ac}(2\tau - u)$. We then have an inversion relation about the point P.

The restriction $0 < \lambda < \ln|1/q|$ implies from (6.20) that $0 < \mu < \pi/2$, which is the interval considered in Section 5. We see that our results must fail in the limit $q^2 \to -1$ when $\pi/3 < \mu < \pi/2$, since then the distribution function $g(y)$ in (5.13) is negative. In any case, we should obtain the Lieb–Sutherland results[20,21] for the six-vertex model, not the corresponding SDP results.

The resolution of this paradox lies in the fact that the eight-vertex model results (6.9) have not been obtained with the same degree of rigor that has been applied to the Ising and six-vertex models. All the methods used[1,6,29] have consisted of obtaining equations for κ, and then finding solutions of these equations. [This is quite different from the Ising model, where one can directly write down tractable expressions for Z for a finite lattice, and then explicitly take the thermodynamic limit (2.4).] Various limits have been interchanged without justification, but more importantly the solution has been chosen to be the one that is correct for small $|q|$ and large λ. This means that (6.9) is undoubtedly exact for sufficiently small $|q|$, but it is conceivable that as $|q|$ is increased there may come a point at which one should change from one solution to another. [In rather the same way as one changes from (6.9a) to (6.9b) when u crosses λ.]

For q and λ real there seems to be no reason to suppose this occurs. However, for q^2 negative and u in domain (ii), it must occur for the eight-vertex model with cyclic boundary conditions; otherwise one does not get the proper six-vertex result in the limit $q^2 \to -1$.

Even so, it is still very interesting that (6.9) should instead give the SDP model results (for $0 < \mu < \pi/3$) in this limit. Presumably this solution corresponds to a six-vertex model with the free boundary conditions (2.26) appropriate to the equivalent Potts model.[19]

6.2. Hard-Hexagon Model

This model is defined in (3.1), (3.2), and (3.17)–(3.20). Here I shall take Δ_h to be real, and Δ_c to be the critical value of Δ_h given by (3.21), i.e., $\Delta_c = [\frac{1}{2}(11 + 5\sqrt{5}\,]^{-1/2} = 0.30028\ldots$. We have to distinguish the cases $|\Delta| > \Delta_c$ and $|\Delta| < \Delta_c$.

$|\Delta| > \Delta_c$. In this case we use the parametrization (28a) of Ref. 3. Replacing x, w therein by $-e^{-\lambda}$, e^{2u}, we can write this as

$$\begin{aligned}
\omega_1 &= \rho_0 e^{-2u}\phi_1(2\lambda + u)/\phi_1(2\lambda)\\
\omega_2 &= \rho_0 e^{\lambda}\phi_1(u)/\left[\phi_1(\lambda)\phi_1(2\lambda)\right]^{1/2}\\
\omega_3 &= \rho_0\phi_1(\lambda - u)/\phi_1(\lambda)\\
\omega_4 &= \rho_0 e^{2u}\phi_1(2\lambda - u)/\phi_1(2\lambda)\\
\omega_5 &= \rho_0 e^{-2u}\phi_1(\lambda + u)/\phi_1(\lambda)\\
\Delta_h &= e^{3\lambda}\left[\phi_1(\lambda)/\phi_1(2\lambda)\right]^{5/2}
\end{aligned} \tag{6.22}$$

Here $\phi_1(u)$ is the function $\phi_1(u,q)$ defined by (6.3) and (6.4) with

$$q^2 = -e^{-5\lambda} \tag{6.23}$$

We regard ρ_0 and λ as constants, and u as a variable. The star–triangle relation (2.49) is satisfied for all complex numbers u and u'. As with the eight-vertex model, the inversion relation (2.50) is also satisfied, provided only that the function $\sinh u$ therein is replaced by $\phi_1(u,q)$. Furthermore, replacing u by $\lambda - u$ in (6.22) merely interchanges ω_2 with ω_3, and ω_4 with ω_5 [apart from factors that can be absorbed into the irrelevant variable t in (3.19)]. These interchanges correspond to rotating the lattice through 90°, so again we have the symmetry property (2.39).

It follows that $\kappa(u)$ must satisfy precisely the same inversion relations as those of the eight-vertex model, namely, (6.6). It also has the same periodicity properties (6.7), is analytic in the vertical strips shown in Fig. 4, and can be analytically continued across the strip boundaries.

It follows that $\kappa(u)$ must be the same as for the eight-vertex model, and indeed on substituting into (6.9) the value (6.23) of q^2, we obtain

(i) $0 < \mathrm{Re}(u) < \lambda$:

$$\kappa(u) = \frac{\omega_4\omega_5}{\omega_1} e^u \frac{\phi_2\left(\frac{1}{2}\lambda - u, e^{-2\lambda}\right)}{\phi_2\left(\frac{1}{2}\lambda + u, e^{-2\lambda}\right)} \tag{6.24a}$$

(ii) $\lambda < \mathrm{Re}(u) < 5\lambda/2$:

$$\kappa(u) = \frac{\omega_4\omega_5}{\omega_1} e^{2\lambda - 2u} \frac{\phi_1(u, e^{-3\lambda})\phi_2\left(\frac{1}{2}\lambda - u, e^{-3\lambda}\right)}{\phi_1(2\lambda - u, e^{-3\lambda})\phi_2(3\lambda/2 - u, e^{-3\lambda})} \tag{6.24b}$$

These are precisely the regime IV and regime I results for the hard-hexagon model. In these cases the hard hexagon model is therefore equivalent to an eight-vertex model, in which q^2 is negative and is related to λ by (6.23).

The critical case $\Delta_h = \Delta_c$ is obtained by letting $\lambda \to 0$, while keeping

$$v = \pi u / 5\lambda \tag{6.25}$$

fixed. Using the periodicity relation (6.7), we obtain the results (5.18). We thus regain the equivalence discussed in Section 5 between the critical hard-hexagon and SDP models.

$|\Delta_h| < \Delta_c$. Now we use the parametrization (28b) of Ref. 3, with x, w therein replaced by $e^{-2\lambda}$, e^{-2u}. This gives

$$\begin{aligned}
\omega_1 &= \rho_0 e^{-u}\phi_1(2\lambda + u)/\phi_1(2\lambda) \\
\omega_2 &= \rho_0 e^{-u+\lambda/2}\phi_1(u)/\left[\phi_1(\lambda)\phi_1(2\lambda)\right]^{1/2} \\
\omega_3 &= \rho_0 e^{u}\phi_1(\lambda - u)/\phi_1(\lambda) \\
\omega_4 &= \rho_0 e^{-u}\phi_1(2\lambda - u)/\phi_1(2\lambda) \\
\omega_5 &= \rho_0 e^{u}\phi_1(\lambda + u)/\phi_1(\lambda) \\
\Delta_h &= -e^{3\lambda/2}\left[\phi_1(\lambda)/\phi_1(2\lambda)\right]^{5/2}
\end{aligned} \tag{6.26}$$

where here $\phi_1(u)$ is defined by (6.3) and (6.4) with

$$q^2 = e^{-10\lambda} \tag{6.27}$$

Again the star–triangle relation (2.49), the inversion relation (2.50), and the symmetry relation (2.39) are satisfied, provided only that $\sinh u$ is replaced by $\phi_1(u,q)$. However, this time we do not obtain the result (6.9). Instead, $\kappa(u)$ is given by

(i) $0 < \mathrm{Re}(u) < \lambda$:

$$\kappa(u) = \omega_4\omega_5/\omega_1 \tag{6.28a}$$

(ii) $\lambda < \mathrm{Re}(u) < 2\tfrac{1}{2}\lambda$:

$$\kappa(u) = \frac{\omega_4\omega_5}{\omega_1} e^{4(\lambda-u)/3} \frac{\phi_1(u,e^{-3\lambda})}{\phi_1(2\lambda-u,e^{-3\lambda})} \tag{6.28b}$$

(iii) $2\tfrac{1}{2}\lambda < \mathrm{Re}(u) < 3\tfrac{1}{2}\lambda$:

$$\kappa(u) = \frac{\omega_4\omega_5}{\omega_1} e^{3\lambda/4} \frac{\phi_1(u-3\lambda,e^{-2\lambda})}{\phi_4(u-3\lambda,e^{-2\lambda})} \tag{6.28c}$$

(iv) $3\tfrac{1}{2}\lambda < \mathrm{Re}(u) < 5\lambda$:

$$\kappa(u) = \frac{\omega_4\omega_5}{\omega_1} e^{4(u-5\lambda)/3} \frac{\phi_1(6\lambda-u,e^{-3\lambda})}{\phi_1(4\lambda-u,e^{-3\lambda})} \tag{6.28d}$$

The apparent poles and zeros in these expressions cancel one another: for real u, $\kappa(u)$ is continuous and positive. The form (i) is the "regime III" result of the hard-hexagon model 3; (ii) and (iv) are the results for regimes VI and II; (iii) is the result for a previously unreported nonphysical ordered regime, in which every other site on every other row is preferentially occupied.

These results also reduce to (5.18) in the critical limit, when $\lambda \to 0$ while v in (6.25) is held fixed. This means that κ is continuous across the critical line $|\Delta| = \Delta_c$. It also means that in this limit the model is equivalent to an SDP model, rather than a six-vertex model.

Even for nonzero λ, this model bears the same relationship to the eight-vertex model as the SDP model does to the six-vertex model. Instead of having two domains, as in Fig. 2a, there are four domains, as in Fig. 3a. Lying on the boundaries are the true inversion points $u = 0, \lambda, -\ln q$, and the virtual inversion points $-\tfrac{1}{2}\ln q, \lambda - \tfrac{1}{2}\ln q$. The corresponding inversion relations are satisfied and define $\kappa(u)$, but since these properties differ from those of the eight-vertex model, we obtain different answers [except in domain (i), corresponding to the hard-hexagon regime III].

6.3. "SDP-like" Eight Vertex Model

Suppose we generalize the properties mentioned in the last paragraph, and simply define $\kappa(u)$ to satisfy the periodicity relation (6.7b), the true inversion relations (6.6) and (6.10), and the virtual inversion relations

$$\kappa(u)\kappa_{ac}(\tau - u) = -\rho_0^2 q^{-1} e^{-2u-\lambda}\phi_1(\lambda - u)\phi_1(\lambda + u)/\phi_1^2(\lambda) \quad (6.29a)$$

$$\kappa(u)\kappa_{ac}(\tau + 2\lambda - u) = -\rho_0^2 q^{-1} e^{3\lambda-2u}\phi_1(u)\phi_1(2\lambda - u)/\phi_1^2(\lambda) \quad (6.29b)$$

where q and λ are independent real positive parameters, τ is related to q by (6.11), i.e.,

$$q = e^{-\tau} \quad (6.30)$$

and $0 < \lambda < \tau$. [All five inversion relations can be obtained from (6.6a) by blindly using (anti-) periodicity and the symmetry property (2.39), as though $\kappa_{ac}(u)$ were a single-valued function.]

Suppose also that $\kappa'(u)/\kappa(u)$ is periodic of period πi, and that $\ln\kappa(u)$ is analytic in the four vertical strips between the inversion points, and can be analytically continued a short distance across the strip boundaries. (As usual, κ_{ac} means this analytic continuation.) Then κ is defined by these properties. Like the κ of the true eight-vertex model, considered as a function of u and λ (ρ_0 and τ being held fixed) it satisfies the symmetry relation

$$\kappa(u,\lambda) = e^{u-\lambda}\kappa(\tau - u, \tau - \lambda) \quad (6.31)$$

so without loss of generality we can require that $0 < \lambda < \frac{1}{2}\tau$. Then it follows that

(i) $0 < \mathrm{Re}(u) < \lambda$:

$$\ln\kappa = \ln\rho_0 + S(e^{-2u}, e^{-2\lambda}, q^2) \quad (6.32a)$$

(ii) $\lambda < \mathrm{Re}(u) < \frac{1}{2}\tau$:

$$\ln\kappa = \ln\rho_0 + \frac{(u-\lambda)(\tau - 3\lambda)}{\tau - 2\lambda} + 2\sum_{n=1}^{\infty} \frac{\sinh n(u-\lambda)\,\phi(\pi\lambda/\tau, \pi u/\tau, n\tau/\pi)}{n \sinh n\tau \sinh n(\tau - 2\lambda)} \quad (6.32b)$$

(iii) $\frac{1}{2}\tau < \mathrm{Re}(u) < \lambda + \frac{1}{2}\tau$:

$$\ln\kappa = \ln\rho_0 = u - \frac{3\lambda}{2} + \sum_{n=1}^{\infty} \frac{\psi(\pi\lambda/\tau, \pi u/\tau, n\tau/\pi)}{n\sinh n\tau \cosh n\lambda} \quad (6.32c)$$

(iv) $\lambda + \frac{1}{2}\tau < \mathrm{Re}(u) < \tau$:

$$\ln\kappa = \ln\rho_0 + \frac{(u-2\lambda)(\tau-\lambda)}{\tau - 2\lambda} + 2\sum_{n=1}^{\infty} \frac{\sinh n(\tau - u)\,\phi[\pi\lambda/\tau, \pi(\lambda + \tau - u)/\tau, n\tau/\pi]}{n\sinh n\tau \sinh n(\tau - 2\lambda)} \quad (6.32d)$$

Here $S(w, y, p)$ is the function defined by (6.8); $\phi(\mu, v, t)$ and $\psi(\mu, v, t)$ are the functions defined by (5.12c) and (5.12e); $|\kappa(u)|$ is continuous across the domain boundaries.

These results can be combined into the single formula (true for all complex numbers u):

$$\ln \kappa = \int_0^{\pi} \ln\left[\frac{\rho_0^2 \phi_1(u-iy)\,\phi_1(u+iy-\lambda)}{\phi_1(iy)\,\phi_1(\lambda-iy)} \right] g(y)\,dy$$
$$+ \int_0^{\pi} \ln\left[\frac{\rho_0^2 \phi_1(u-\frac{1}{2}\tau-iy)\,\phi_1(u+iy-\lambda-\frac{1}{2}\tau)}{\phi_1(\frac{1}{2}\tau+iy)\,\phi_1(\lambda+\frac{1}{2}\tau-iy)} \right] h(y)\,dy \qquad (6.33)$$

where

$$g(y) = \frac{1}{2\pi}\left[\frac{\tau-3\lambda}{\tau-2\lambda} + 2\sum_{n=1}^{\infty} \frac{\cos 2ny \,\sinh n(\tau-3\lambda)}{\cosh n\lambda \,\sinh n(\tau-2\lambda)} \right] \qquad (6.34a)$$

$$h(y) = \frac{1}{2\pi}\left[\frac{\lambda}{\tau-2\lambda} + 2\sum_{n=1}^{\infty} \frac{\cos 2ny \,\sinh n\lambda}{\cosh n\lambda \,\sinh n(\tau-2\lambda)} \right] \qquad (6.34b)$$

These equations reduce to the SDP results (5.12)–(5.14) in the limit when $\tau, \lambda, u \to 0$, the ratios $\mu = \pi\lambda/\tau$, $v = \pi u/\tau$ remaining fixed. They also reduce to the $|\Delta_h| < \Delta_c$ results of the hard-hexagon model when $\tau = 5\lambda$. The functions $g(y)$, $h(y)$ are then the distribution functions for the zeros of $Z(u)$ along the lines $\mathrm{Re}(u) = 0$ and $\mathrm{Re}(u) = \frac{1}{2}\tau$, respectively.

These are the only two realizations I know of the expressions (6.32)–(6.34) for κ, but it is conceivable that others may yet be found. Perhaps they correspond to an eight-vertex model with some particular noncyclic boundary conditions, just as the SDP model corresponds to such a six-vertex model. Alternatively, it may be possible to regard these results as some kind of continuation (continuous, but not analytic) of the eight-vertex model results across the SDP case $q^2 = -1$. (The hard-hexagon regime II and III results can be regarded as such a continuation of those for regimes I and IV.)

If $g(y)$ and $h(y)$ continue to be distribution functions, then they must of course be nonnegative. [The same conclusion follows if one requires simply that $\kappa(u)$ be given by (2.4), where $Z(u)$ is entire: any closed contour integral of $(2\pi i)^{-1} N\kappa'(u)/\kappa(u)$ then simply counts the number of zeros inside the contour, so cannot be negative.] From (6.34), a necessary condition for this is that

$$0 < \lambda < \tau/3 \qquad (6.35)$$

which corresponds to the fact that the SDP results (5.12) apply only for $0 < \mu < \pi/3$.

Note that $\kappa(u)$ is not strictly periodic of period πi within the domains (ii) and (iv), due to the second terms in (6.32b) and (6.32d). For arbitrary values of λ/τ it is hard to see how this can arise, but for the regime II hard-hexagon model, where $\tau = 5\lambda$, these terms contribute a factor $\exp(2u/3)$ to κ. This is a three-valued function of e^{2u}, and arises because the system has triangular ordering, every third site being preferentially occupied.

7. SUMMARY

We have considered the zero-field six- and eight-vertex models, the generalized hard hexagon model, and the self-dual Potts model. Each of these has local transfer matrices that satisfy the star–triangle relation (2.6). With an appropriate parametrization, this can be written more explicitly as (2.49). The matrices then satisfy the inversion relation (2.50), except that in general the function $\sinh u$ therein is replaced by the elliptic theta function $\phi_1(u, q)$. Using the rotation symmetry (2.39), it follows that[6,7] the partition function per site $\kappa(u)$ satisfies (2.40) or (2.47), or in general (6.6). The eight-vertex model is the most general of these models: it includes all possible choices of the various parameters (ρ_0, τ, λ, and u).

Together with some basic analyticity and periodicity properties (which are closely linked[7] with the star–triangle relation), these equations determine $\kappa(u)$. It might therefore be thought that all the models have the same κ. This is true for $0 < u < \lambda$ (or $0 < v < \mu$), corresponding to the ferromagnetic cases of the SDP model, and the "regimes III and IV" cases of the hard-hexagon model. However, outside this interval (antiferromagnetic or nonphysical SDP, and regimes I and II of hard hexagons) it is not necessarily so.

The difference arises because $\kappa(u)$ has two possible types of analytic behavior. In both cases it is piecewise analytic in vertical strips, but in one case (the vertex models, and regimes I and IV of hard hexagons) there are just two such strips, while in the other case (the SDP model and hard-hexagon regimes II and III) there are four strips.

For the six-vertex, SDP and critical hard-hexagon models, we have a stronger equivalence. Their local transfer matrices X_j have the form (2.36), where the matrices $U_1, \ldots, U_n$ are independent of the variable u, and satisfy (2.19). This means that the matrices generate an algebra. If one defines the partition function Z by (2.26) (or by any other definition that depends only on the properties of this algebra), then Z must be the same (even for a finite lattice) for all three models, provided they have the same values of ρ_0, λ, and u. This is a special case of the equivalence found by Temperley and Lieb[9] for the six-vertex and Potts models.

This equivalence is satisfied for the cases corresponding to $0 < u < \lambda$ or $0 < v < \mu$, namely, the six-vertex model with $\Delta < 0$, the SDP model with ferromagnetic interactions, and the hard-hexagon model on the critical line between regimes III and IV. It is not satisfied outside this interval: the six-vertex model has "two-strip" behavior, while the SDP and critical hard-hexagon models have four strips. The source of this failure appears to be the sensitivity of the six-vertex model to boundary conditions.

Even so, it is always true that the critical hard-hexagon model is equivalent to (i.e., has the same κ as) the self-dual Potts model with $q_p = \frac{1}{2}(3+\sqrt{5}) = 2.618\ldots$, and this is perhaps the most interesting result of this paper. As I remarked at the end of Section 1, this equivalence is true only at criticality, and does not affect the argument[11,30] that the critical exponents of the hard-hexagon model should be those of the $q_p = 3$ Potts model.

Two immediate problems that emerge are: what is κ for the self-dual Potts model with $q_p > 4$ inside domains (ii) and (iii) of Fig. 1; and what is κ for the antiferromagnetic self-dual Potts model with $0 < q_p < 1$? These problems may be "unphysical," but their solution would help us understand κ as a function of u and λ, and hence the role of the matrix inversion relations for these cases. It would also be interesting to check whether the vertex models considered by Stroganov[4] and Schultz[5] are equivalent to the six-vertex model, or to the SDP model.

NOTE ADDED IN PROOF

The critical antiferromagnetic square lattice Potts model has now been solved, [31] using the inversion relation method. A much simplified form of (5.12) is given in eq. (30) of that paper: corresponding simplifications can be made of (6.32).

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